Around Tokuyama’s formula

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Definition

A strict Gelfand-Tsetlin (GT) pattern is a triangular array of non-negative integers

\[
\begin{bmatrix}
a_{1,1} & a_{1,2} & \cdots & a_{1,n} \\
a_{2,2} & \cdots & a_{2,n} \\
& \cdots & \cdots \\
a_{n,n}
\end{bmatrix}
\]

satisfying \( a_{i,j} \geq a_{i+1,j+1} \geq a_{i,j+1} \) (*) and \( a_{i,j} > a_{i,j+1} \) (**).

Examples:

\[
\begin{bmatrix}
2 & 0 & 3 & 1 & 0 \\
1 & 2 & 1 & 0 \end{bmatrix}
\]

\[
\begin{bmatrix}
4 & 3 & 2 & 0 \\
4 & 3 & 2 \\
3 & 2 \\
2
\end{bmatrix}
\]
Combinatorial data of GT patterns

- **left-leaning** if it equals the entry above on the left.\n  \[ l(T) = \# \text{ left-leaning entries} \]
- **right-leaning** if it equals the entry above on the right.\n  \[ r(T) = \# \text{ right-leaning entries} \]
- **generic** if it is neither left-, nor right-leaning.\n  \[ g(T) = \# \text{ generic entries} \]

\[ \begin{array}{cccccc}
3 & 1 & 0 & 4 & 3 & 2 \\
\triangle & 1 & \square & \square & \square & \triangle \\
1 & \bigtriangleup & 4 & 3 & 2 & \bigtriangleup \\
\end{array} \]

\[ l(T) = 1, r(T) = 1, g(T) = 1; \quad l(T) = 2, r(T) = 3, g(T) = 1 \]
Weights of strict GT patterns

Definition
Let $T$ be a strict GT pattern. Then the weight of $T$ is

$$\text{weight}(T) = (1 + t)^{g(T)} t^{l(T)} \prod_{i=1}^{n} z_i^{m_i(T)}$$

where $m_i(T) = \sum$ entries on ith row $-$ $\sum$ entries on i+1st row

$$\text{weight} \begin{pmatrix} 2 & 0 \\ 1 & 0 \end{pmatrix} = (1 + t)^{g(T)} t^{l(T)} z_1^{m_1} z_2^{m_2} = (1 + t)z_1z_2$$

$$\text{weight} \begin{pmatrix} 3 & 1 & 0 \\ 2 & 1 & 0 \\ 1 & 1 & 0 \end{pmatrix} = (1+t)^{g(T)} t^{l(T)} z_1^{m_1} z_2^{m_2} z_3^{m_3} = (1+t)tz_1^2z_2z_3$$
Tokuyama’s formula

**Theorem (Tokuyama)**

Let \( \lambda = (\lambda_1, \ldots, \lambda_n) \) be a partition (weakly decreasing sequence of non-negative integers) and \( \rho = (n - 1, n - 2, \ldots, 1, 0) \). Then,

\[
\sum_{T \in SGT(\lambda+\rho)} \text{weight}(T) = \prod_{i<j}(z_i + tz_j)s_\lambda(z_1, \ldots, z_n)
\]

where \( \text{weight}(T) = (t + 1)^{g(T)}t^{l(T)} \prod_{i=1}^{n} z_i^{m_i(T)} \).

- *left-hands side is a sum over all strict GT patterns with top row \( \lambda + \rho \).*
- *right-hand side is product times Schur polynomial \( s_\lambda \) (characters of polynomial irreducible representations of the general linear groups).*
Example of Tokuyama’s formula

\[ \sum_{T \in SGT(\lambda+\rho)} \text{weight}(T) = \prod_{i<j} (z_i + tz_j) s_{\lambda}(z_1, \ldots, z_n) \]

Let \( \lambda = (1,0); \rho = (1,0); \lambda + \rho = (2,0) \). The patterns are

\[
\begin{array}{cccc}
2 & 0 & 2 & 0 \\
2 & 1 & 2 & 0 \\
\end{array}
\]

\[ tz_2^2 \quad (1 + t)z_1z_2 \quad z_1^2 \]

Tokuyama’s formula becomes

\[ tz_2^2 + (t + 1)z_1z_2 + z_1^2 = (z_1 + tz_2)(z_1 + z_2) \]

where \( s_{(1,0)}(z_1, z_2) = z_1 + z_2 \).
Existing Proofs of Tokuyama’s Formula

- A direct proof by induction by Bump.
- A combinatorial proof by Okada.
- A proof by lattice models by Brubaker, Bump, and Friedberg.
Bump’s Direct Proof

We recall Tokuyama’s formula

\[ \sum_{T \in \{\text{GT patterns}\}} \text{weight}(T) = \prod_{i < j} (z_i + t z_j) s_{\lambda}(z_1, \ldots, z_n) \]

Define the functions \( L \) and \( R \) to be the left and right sides of Tokuyama’s formula, parametrized by the partition \( \lambda \) of length \( n \):

\[
L(\lambda; t; z_1, \ldots, z_n) = \sum_{T \in \{\text{GT patterns}\}} \text{weight}(T) \\
R(\lambda; t; z_1, \ldots, z_n) = \prod_{i < j} (z_i + t z_j) s_{\lambda}(z_1, \ldots, z_n).
\]

We use induction on the length of \( \lambda \) to show that \( L = R \).
Why Induction?

Strict Gelfand-Tsetlin patterns naturally lend themselves to induction:

```
  6 3 1
  5 3
  4
```
Why Induction?

Strict Gelfand-Tsetlin patterns naturally lend themselves to induction:

\[
\begin{array}{ccc}
6 & 3 & 1 \\
5 & 3 & \\
4 & \\
\end{array}
\]

This gives us a way to inductively compute the weight:

\[
\text{weight } \begin{pmatrix} 6 & 5 & 3 & 1 \\ 3 & 4 & \end{pmatrix} = (t + 1)tz_1^2 \cdot \text{weight } \begin{pmatrix} 5 & 4 & 3 \end{pmatrix}
\]
Drawbacks

While $L$, which is a sum over GT-patterns, works well with induction, manipulating the representation-theoretic $R$ requires heavy machinery:

- **Branching Rules:**

  \[ s_\lambda(z_1, \ldots, z_n) = \sum_\nu z_1^{\lambda \setminus \nu} s_\nu(z_2, \ldots, z_n) \]

  where $\nu$ interleaves $\lambda$, i.e., $\lambda_1 \geq \nu_1 \geq \lambda_2 \geq \nu_2 \geq \cdots$

- **Pieri’s Formula:**

  \[ e_k s_\nu(z_2, \ldots, z_n) = \sum_\mu s_\mu(z_2, \ldots, z_n) \]

  where $e_k = \sum_{i_1 < \cdots < i_k} z_{i_1} \cdots z_{i_k}$ and $\mu$ are partitions such that $\nu \setminus \mu$ is a horizontal strip
Okada’s Combinatorial Proof

Okada’s proof uses a chain of complicated combinatorial objects to reframe the question of Tokuyama’s Formula.

\[
\text{Diagonal-Strict Shifted Plane Partitions} \xrightarrow{\text{weight-preserving bijection}} \text{Gelfand-Tsetlin Patterns} \xrightarrow{\text{lifting argument}} (A,B)\text{-Partially Strict Shifted Plane Partitions} \xrightarrow{\text{weight-preserving bijection}} \text{Non-Intersecting Lattice Path Systems}
\]
Lattice Paths and the LGV Lemma

The Lindström-Gessel-Viennot Lemma allows us to turn weighted sums over lattice path systems into a determinant of weighted sums over individual paths.

We can explicitly evaluate these weighted sums over individual paths, and then use a determinant expression for Schur polynomials.

**Major Downside:** the determinant expression for Schur polynomials that Okada uses is difficult to analogize.
We introduce the Gamma ice model. Given a partition $\lambda = (\lambda_1, \ldots, \lambda_n)$ we fill a grid of circles with either $+1$ or $-1$. Example for $\lambda = (5, 2, 0)$. 

![Gamma ice example](image-url)
Gamma ice

One way to fill the previous example for $\lambda = (5, 2, 0)$:

It turns out that there is a bijection between Gamma ice configurations and Gelfand-Tsetlin patterns. The corresponding Gelfand-Tsetlin pattern for the example Gamma ice is

\[
\begin{pmatrix}
5 & 2 & 0 \\
3 & 0 \\
3 & 3 \\
\end{pmatrix}
\]
Why do we use this model?

We are interested in evaluating

\[
\sum_{T \in \{\text{GT patterns}\}} \text{weight}(T) = \sum_{x \in \{\text{Gamma Ice config.}\}} \text{weight}(x)
\]

Framing the problem in terms of Gamma Ice configurations allows us to use a tool called the Yang-Baxter equation.
Yang-Baxter equation

\[ \sum_{\gamma, \mu, \nu} g = \sum_{\delta, \phi, \psi} \]

**Downside:** There is not a clear motivation for using the Gamma Ice model. Also, the methods seem to be ad-hoc to this setting and not generalizable.
Overview of Our Contribution

Our group’s contribution to the standard Tokuyama’s formula consists of two novel proofs which both rely on the same principle:

\[
\begin{align*}
&\text{Begin with combinatorial formula} \\
&\sum_T \text{weight}(T) \\
&\text{Pair GT-patterns to extract factors of the form } (z_i + tz_j) \\
&\text{Specialize parameter } t \text{ to obtain } \prod(z_i + tz_j)s_\lambda(z)
\end{align*}
\]

This process avoids the major downsides of the three proofs described previously.
Our New Direct Proof

We use the same $L$ and $R$ as in Bump's proof

$$L(\lambda; t; z_1, \ldots, z_n) = \sum_{T \in \{GT \text{ patterns}\}} \text{weight}(T)$$

$$R(\lambda; t; z_1, \ldots, z_n) = \prod_{i<j} (z_i + tz_j) s_{\lambda}(z_1, \ldots, z_n).$$

we can directly show that $L = R$ if we can show that we can factor out $\prod_{i<j} (z_i + tz_j)$ from the sum in $L$. 
Pairing

To evaluate the sum, we want to factor out the terms \((z_1 + tz_j)\) for each \(j\). Example of pairing for \(\lambda = (6, 3, 1)\)

\[
\begin{array}{c}
\{\begin{array}{ccc}
6 & 3 & 1 \\
6 & 3 & 1
\end{array}\} \leftarrow (z_1 + tz_2) \rightarrow \{\begin{array}{ccc}
5 & 3 & 1 \\
4 & 3 & 1
\end{array}\} \leftarrow (z_1 + tz_2) \rightarrow \{\begin{array}{ccc}
5 & 3 & 1 \\
4 & 3 & 1
\end{array}\} \leftarrow (z_1 + tz_2) \rightarrow \{\begin{array}{ccc}
6 & 3 & 1 \\
6 & 3 & 1
\end{array}\} \\
\end{array}
\]
Pairing

\[
t^2 z_1^6 z_2^3 z_3^2 \leftarrow (z_1 + tz_2) \rightarrow t z_1^2 z_2^5 z_3^3
\]
\[
t^2 z_1^3 z_2^5 z_3^2 \leftarrow (z_1 + tz_2) \rightarrow t z_1^3 z_2^4 z_3^3
\]
\[
t^2 z_1^3 z_2^3 z_3^3 \leftarrow (z_1 + tz_2) \rightarrow t z_1^4 z_2^3 z_3^3
\]

\[
t z_1^2 z_2^6 z_3^2 \leftarrow (z_1 + tz_3) \rightarrow z_1^2 z_2^5 z_3^2
\]
\[
t z_1^3 z_2^5 z_3^2 \leftarrow (z_1 + tz_3) \rightarrow z_1^3 z_2^4 z_3^2
\]
\[
t z_1^3 z_2^3 z_3^3 \leftarrow (z_1 + tz_3) \rightarrow z_1^4 z_2^3 z_3^3
\]

\[
t z_1^4 z_2^4 z_3^2 \leftarrow (z_1 + tz_3) \rightarrow z_1^5 z_2^3 z_3^2
\]
\[
t z_1^4 z_2^4 z_3^2 \leftarrow (z_1 + tz_3) \rightarrow z_1^5 z_2^3 z_3^2
\]
\[
t z_1^5 z_2^4 z_3^2 \leftarrow (z_1 + tz_3) \rightarrow z_1^6 z_2^3 z_3^3
\]

\[
tz_2 (z_1^4 z_2^4 z_3) + z_1 (z_1^4 z_2^4 z_3) = (tz_2 + z_1) z_1^4 z_2^4 z_3
\]
Primed strict GT-patterns

Take a strict GT-pattern.

- Put a prime on left-leaning terms.
- Split each generic entry into two entries - one with a prime and one without. We get \( 2^g(T) \) Primed strict GT (PSGT) patterns, weighted as so:

\[
w(T) = t\# \text{ primed} \prod_{i=1}^{n} z_{m_i}(t)
\]

\[
\begin{align*}
2 & \quad 0 & \quad 2 & \quad 0 & \quad 2 & \quad 0 & \quad 2 & \quad 0 \\
2' & \quad 1' & \quad 2 & \quad 1 & \quad 2 & \quad 0 \\
&t z_2^2 & \quad t z_1 z_2 & \quad z_1 z_2 & \quad z_1^2
\end{align*}
\]
Combinatorial Tokuyama

Tokuyama’s formula:

\[
\sum_{T \in SGT(\lambda + \rho)} (1 + t)^{g(t)} t^{l(t)} \prod_{i=1}^{n} z_i^{m_i(t)} = \prod_{i < j} (z_i + tz_j) s_\lambda(z_1, \ldots, z_n)
\]

Our Combinatorial Tokuyama formula:

\[
\sum_{T \in PSGT(\lambda + \rho)} t^{\# \text{ primed}} \prod_{i=1}^{n} z_i^{m_i(T)} = \prod_{i < j} (z_i + tz_j) s_\lambda(z_1, \ldots, z_n)
\]
An example of the two formulas

Consider $\lambda = (2, 1)$, so $\lambda + \rho = (2, 1) + (1, 0) = (3, 1)$. Then:

<table>
<thead>
<tr>
<th>$\text{SGT}(\lambda + \rho)$</th>
<th>Weight</th>
<th>$\text{PSGT}(\lambda + \rho)$</th>
<th>Weight</th>
</tr>
</thead>
<tbody>
<tr>
<td>$3$</td>
<td>$1$</td>
<td>$tz_1z_2^3$</td>
<td>$3$</td>
</tr>
<tr>
<td>$3'$</td>
<td></td>
<td>$tz_1z_2^3$</td>
<td></td>
</tr>
<tr>
<td>$2$</td>
<td>$1$</td>
<td>$(1 + t)z_1^2z_2^2$</td>
<td>$3$</td>
</tr>
<tr>
<td>$2'$</td>
<td></td>
<td>$z_1^2z_2^2$</td>
<td></td>
</tr>
<tr>
<td>$1$</td>
<td>$1$</td>
<td>$z_1^3z_2$</td>
<td>$3$</td>
</tr>
<tr>
<td>$1$</td>
<td></td>
<td>$z_1^3z_2$</td>
<td>$1$</td>
</tr>
</tbody>
</table>

In either case the sum of all the weights resolves to $tz_1z_2^3 + (1 + t)z_1^2z_2^2 + z_1^3z_2 = (z_1 + tz_2)(z_1z_2(z_1 + z_2))$. 
Pairing primed patterns

We partition the set of primed Gelfand-Tsetlin patterns into pairs such that each pair is divisible by \((z_1 + tz_2)\). This proves that \((z_1 + tz_2)\) divides

\[
\sum_{T \in PSGT(\lambda + \rho)} t^\# \text{ primed} \prod_{i=1}^n z_i^{m_i(T)}
\]

\[
\begin{array}{cccc}
2 & 0 & 2 & 0 \\
2' & 1 \\
tz_2^2 & + & z_1z_2 = (z_1 + tz_2)z_2 \\
2 & 0 & 2 & 0 \\
1' & 0 \\
tz_1z_2 & + & z_1^2 = (z_1 + tz_2)z_1
\end{array}
\]
The pairing

Pairing Algorithm: Look at the second row of a primed strict GT pattern. Take the first entry from left to right for which we can either:

- remove a prime and decrease the entry by 1, or
- put a prime and increase the entry by 1

\[
\begin{array}{cccc}
3 & 1 & 0 & 3 \\
3' & 1' & 2 & 1' \\
1 & 1 \\
\end{array}
\]

\[t^2 z_2^3 z_3 + t z_1 z_2^2 z_3 = (z_1 + tz_2) tz_2^2 z_3\]

\[
\begin{array}{cccc}
3 & 1 & 0 & 3 \\
2' & 1' & 2' & 0 \\
1 & 1 \\
\end{array}
\]

\[t^2 z_1 z_2^2 z_3 + t z_1^2 z_2 z_3 = (z_1 + tz_2) tz_1 z_2 z_3\]
Concluding the Argument

Once we demonstrate the weighted generating function

$$\sum_{T \in PSGT(\lambda + \rho)} t^\# \text{ primed} \prod_{i=1}^{n} z_i^{m_i(T)}$$

is divisible by $(z_1 + tz_2)$, we use essentially symmetric arguments to prove the same for $(z_i + tz_j)$ for each $i < j$.

All that remains is to show that what is "left over" is $s_\lambda(z_1, \ldots, z_n)$. 
What’s The Point?
Broader Context

Spherical Whittaker Function
\[ W^\circ(g) = \int_U f^\circ(w_0 u g) \psi^{-1}(u) \, du \]

Casselman-Shalika Expression
\[ \prod_{i<j} (z_i + tz_j)s_\lambda(z_1, \ldots, z_n) \]

Tokuyama’s Expression
\[ \sum_{T \in \{GT \text{ patterns}\}} \text{weight}(T) \]

**Figure:** Equivalent expressions with respect to $GL_n$, which has characters $s_\lambda$. 
Broader Context

Spherical Whittaker Function

\[ W^\circ(g) = \int_U f^\circ(w_0ug)\psi^{-1}(u) \, du \]

Casselman-Shalika Expression

\[ z^{-\rho} \prod_{\alpha \in \Phi^+} (1 + tz^\alpha) \chi_\lambda(z_1, \ldots, z_n) \]

Tokuyama’s Expression

\[ \sum_{T \in \{ \text{combinatorial objects} \}} \text{weight}(T) \]

Figure: Equivalent expressions with respect to a reductive group \( G \), which has characters \( \chi_\lambda \)
Symplectic Tokuyama’s Formula

• Using lattice models, Ivanov (2012) proved an analogue to Tokuyama’s formula for the symplectic group $\text{Sp}_{2n}$:

$$\sum_{T \in \{\text{symplectic patterns}\}} \text{weight}(T) = z^{-\rho_C} \prod_{i=1}^{n} (1 + tz_i^2) \prod_{i<j} (1 + t z_i z_j)(1 + t z_i z_j^{-1}) \chi_C^\chi(z_1, \ldots, z_n)$$

where $z^{-\rho_C} = z_1^{-n} z_2^{-n+1} \cdots z_n^{-1}$

• Analogous formulas for orthogonal groups and other reductive groups are not known.
Symplectic Gelfand-Tsetlin Patterns

*Symplectic Gelfand-Tsetlin patterns* are similar to the GT patterns. Schematically, for height of 3 they look as follows.
Symplectic Gelfand-Tsetlin Patterns

We can think about it as interleaving two GT patterns, where an in-betweenness condition between the two GT patterns holds.

Examples

\[
\begin{array}{ccc}
3 & 1 & 6 \\
2 & 1 & 5 \\
1 & & 3 \\
0 & & 2 \\
& & 1 \\
\end{array}
\]
Overview of Our Progress

Tokuyama’s formula for the symplectic group $\text{Sp}_{2n}$:

$$\sum_{T \in \{\text{symplectic patterns}\}} \text{weight}(T) = z^{-\rho_C} \prod_{i=1}^{n} (1 + tz_i^2) \prod_{i<j}(1 + tz_i z_j)(1 + tz_i z_j^{-1}) \chi^C_{\lambda}(z_1, \ldots, z_n)$$

We have made non-trivial progress towards proving the symplectic Tokuyama formula through our novel elementary methods. If we succeed, we will provide the first proof of the symplectic Tokuyama formula that does not use solvable lattice models.
Our Direct Proof Progress

Define \( L \) and \( R \) to be the left and right sides of the symplectic Tokuyama's formula, respectively:

\[
L(\lambda; t; z_1, \ldots, z_n) = \sum_{T \in \{\text{symplectic patterns}\}} \text{weight}(T)
\]

\[
R(\lambda; t; z_1, \ldots, z_n) = z^{-\rho_c} \prod_{i=1}^{n} (1 + tz_i^2) \prod_{i<j}(1 + tz_i z_j)(1 + tz_i z_j^{-1}) \chi^c_{\lambda}(z_1, \ldots, z_n).
\]

We wish to factor out all of the \((1 + tz_i^2)\), \((1 + tz_i z_j)\), and \((1 + tz_i z_j^{-1})\) from the combinatorial sum in \( L \).
Induction on the Symplectic Group

In the inductive step, we pair "partial" patterns having three rows. For instance, with (3, 2, 1) as the top row, we might have:

\[
\begin{array}{ccc}
3 & 2 & 1 \\
3 & 2 & 1 \\
3 & 2 \\
\end{array}
\]

\[
\begin{array}{ccc}
3 & 2 & 1 \\
3 & 2 & 1 \\
3 & 2 \\
\end{array}
\]

\[
t^2 z_1^{-1} z_2^3 z_3^2 \\
t^3 z_1 z_2^3 z_3^2
\]
Induction on the Symplectic Group

In the inductive step, we pair "partial" patterns having three rows. For instance, with $(3, 2, 1)$ as the top row, we might have:

\[
\begin{array}{cccc}
3 & 2 & 1 \\
3 & 2 & 1 \\
3 & 2 & \\
\end{array}
\quad
\begin{array}{cccc}
3 & 2 & 1 \\
3 & 2 & 0 \\
3 & 2 & \\
\end{array}
\]

\[t^2 z_1^{-1} z_2^3 z_3^2 + t^3 z_1 z_2^3 z_3^2 = t^2 z_1^{-1} z_2^3 z_3^2 (1 + tz_1^2)\]
Pairings for \((1 + tz_1^2)\)

The \((1 + tz_1^2)\) pairing is not unique:

\[
\begin{align*}
&\binom{6}{3} 3 1 1 \\
&\begin{pmatrix} 6 & 3 & 1 & 1 \\ 3 & 2 & 0 & 0 & 1 & 0 \end{pmatrix} \\
&\begin{pmatrix} 6 & 3 & 1 & 0 \\ 4 & 1 & 0 \end{pmatrix} \\
\end{align*}
\]

\[
\begin{align*}
&t^6 z_1^4 z_2^3 z_3^1 + t^5 z_1^4 z_2 z_3^1 + t^5 z_1^4 z_2^3 z_3^1 + t^4 z_1^4 z_2^3 z_3^1 \\
&t^5 z_1^4 z_2^3 z_3 \\
&t^4 z_1^4 z_2^3 z_3^1 + t^5 z_1^4 z_2 z_3^1 + t^5 z_1^4 z_2^3 z_3^1 + t^4 z_1^4 z_2^3 z_3^1 \\
&6 \ z_1^4 z_2^3 z_3^1 \\
&\begin{pmatrix} 6 & 4 & 3 & 1 & 0 \\ 3 & 1 & 0 \end{pmatrix} \\
\end{align*}
\]
Pairings for \((1 + tz^2_1)\)

We make the decision based on lexicographic order in the second row:

**Decision Rule:** If a decision exists, pair with the pattern whose second row comes first in lexicographic ordering.

\[
\begin{pmatrix}
6 & 3 & 1 & 0 \\
3 & 3 & 1 & 1
\end{pmatrix}
\]

\[
t^6 z^4_1 z^3_2 z^1_3 + t^5 z^4_1 z^3_2 z^1_3 + t^5 z^4_1 z^3_2 z^1_3 + t^4 z^4_1 z^3_2 z^1_3 - t^5 z^6_1 z^3_2 z^1_3
\]

\[
\begin{pmatrix}
6 & 3 & 1 & 0 \\
3 & 3 & 1 & 1
\end{pmatrix}
\]

\[
t^4 z^4_1 z^3_2 z^1_3 + t^5 z^4_1 z^3_2 z^1_3 + t^5 z^4_1 z^3_2 z^1_3 + t^6 z^4_1 z^3_2 z^1_3
\]

\[
\begin{pmatrix}
6 & 3 & 1 & 0 \\
4 & 1 & 0 & 3 & 1
\end{pmatrix}
\]
Pairings for \((1 + tz_1^2)\)

1. Fix a third row and write all patterns with that third row.

\[
\begin{align*}
    &\begin{array}{c}
        \{3 & \ 1 & \ 0 \\
        1 & \ 1 & \ \}
    \end{array} \\
    \text{and} &\begin{array}{c}
        \{3 & \ 1 & \ 0 \\
        2 & \ 1 & \ \}
    \end{array} \\
    \text{and} &\begin{array}{c}
        \{3 & \ 1 & \ 0 \\
        3 & \ 1 & \ \}
    \end{array} \\
\end{align*}
\]

\[
\begin{align*}
    &t^3z_1^3z_2 \\
    \text{and} &t^2z_1^2z_2 + t^3z_1^2z_2 \\
    \text{and} &t^2z_1^{-1}z_2 + tz_1^{-1}z_2 \\
\end{align*}
\]

\[
\begin{align*}
    &tz_1z_2 + z_1z_2 \\
    \text{and} &z_1^{-3}z_2 \\
\end{align*}
\]
Pairings for \((1 + tz_1^2)\)

2. Make the pairings for the pattern with the smallest second row (in lexicographic order). Follow the decision rule.

\[
\begin{align*}
\{3 & \ 1 & \ 0 \} \\
\{1 & \ 1 & \ 1 \}
\end{align*}
\]

\[
\begin{align*}
\{3 & \ 1 & \ 0 \} \\
\{2 & \ 1 & \ 0 \}
\end{align*}
\]

\[
\begin{align*}
\{3 & \ 1 & \ 0 \} \\
\{3 & \ 1 & \ 1 \}
\end{align*}
\]
Pairings for \((1 + tz_1^2)\)

3. Repeat the procedure, starting with the pattern with the smallest second row that still has unpaired terms, until everything is paired.
Pairings for \((1 + tz_1^2)\)

3. Repeat the procedure, starting with the pattern with the smallest second row that still has unpaired terms, until everything is paired.

\[
\begin{align*}
\{ & 3 & 1 & 0 \\
\{ & 1 & 1 \\
& 1 & 0 \\
\end{align*}
\]

\[
\begin{align*}
& t^3z_1^3z_2 \\
\end{align*}
\]

\[
\begin{align*}
& t^2z_1z_2 + t^3z_1z_2 \\
\end{align*}
\]

\[
\begin{align*}
& t^2z_1^{-1}z_2 + tz_1z_2 \\
\end{align*}
\]

\[
\begin{align*}
\{ & 3 & 1 & 0 \\
\{ & 2 & 1 & 1 \\
& 3 & 1 & 1 \\
\end{align*}
\]
Pairings for \((1 + tz_1^2)\)

4. Once everything is paired, celebrate.

\[
\begin{align*}
\{3 &
\begin{array}{ccc}
1 & 1 & 0 \\
1 & 1 & 1 \\
\end{array} \\
\{3 &
\begin{array}{ccc}
2 & 1 & 0 \\
1 & 1 & 1 \\
\end{array} \\
\{3 &
\begin{array}{ccc}
3 & 1 & 0 \\
1 & 1 & 1 \\
\end{array}
\end{align*}
\]

\[
\begin{align*}
t^3z_1^3z_2 & \quad t^2z_1z_2 + t^3z_1z_2 \\
t^2z_1z_2 + tz_1z_2 & \quad t^2z_1^{-1}z_2 + tz_1^{-1}z_2
\end{align*}
\]

\[
\begin{align*}
tz_1^{-1}z_2 + z_1^{-1}z_2 & \quad z_1^{-3}z_2
\end{align*}
\]
Pairings for \((1 + tz_1^2)\)

Computational evidence suggests that the algorithm is correct. Below is a "small" example. It has also been computationally verified in an example with 1511664 patterns.
Pairings for \((1 + tz_1 z_j)\) and \((1 + tz_1 z_j^{-1})\)

For half of the terms, these pairings are identical to those of the general linear group. Example of pairing for second row \((6, 3, 1)\):

\[
\begin{array}{c}
\{6 & 3 & 1 \} \\
6 & 3 & 1 \\
6 & 3 & 1 \\
\end{array}
\quad \xleftarrow{j = 2} \quad
\begin{array}{c}
\{6 & 3 & 1 \} \\
6 & 3 & 1 \\
5 & 3 & 1 \\
\end{array}
\quad \xleftarrow{j = 2} \quad
\begin{array}{c}
\{6 & 3 & 1 \} \\
6 & 3 & 1 \\
4 & 3 & 1 \\
\end{array}
\quad \xleftarrow{j = 2} \quad
\begin{array}{c}
\{6 & 3 & 1 \} \\
6 & 3 & 1 \\
3 & 3 & 1 \\
\end{array}
\end{array}
\]
**Combinatorial Method**

**Question:** Can we copy our method from the general linear group and apply it to the symplectic group?

- Define primed symplectic patterns and respective weights.

\[
\begin{align*}
3 & \quad 2 \quad 1 \\
3 & \quad 1' \quad 0' \\
2' & \quad 1' \\
1' & \quad 0' \\
1' & \quad 0' \\
1'
\end{align*}
\]

\[\rightarrow t^\# \text{ primed } \prod z_i^{m_i} = t^7 z_1^5 z_2^2 z_3^{-1}\]

- Introduce a Combinatorial Tokuyama formula.

\[
\sum_{T \in \text{primed symplectic patterns}} \text{weight}(T) = z^{-\rho} \prod_{i=1}^{n} (1+tz_i^2) \prod_{i<j} (1+tz_i z_j^{-1})(1+tz_i z_j) \chi_c(z)
\]
Results from the Combinatorial Method

Proposition

We found a pairing algorithm such that the weight of each pair is divisible by the factor $1 + tz_1^2$. It is similar to the pairing algorithm from the general linear group.

\[
t^2 z_1 z_2^{-2} + tz_1^{-1} z_2^{-2} = (1 + tz_1^2)tz_1^{-1} z_2^{-2}
\]
Other factors

- We do not have rigorous pairing algorithms for the factors $1 + tz_1 z_2$ and $1 + tz_1 z_2^{-1}$.
- Main obstacle is that the pairing algorithms require changing multiple entries in the primed symplectic patterns:

$$
\begin{align*}
2 & \quad 1 & 2 & \quad 1 \\
2 & \quad 1 & 2 & \quad 0' \\
2' & \quad 1' & 2 & \quad 1 \\
& \quad 2 & & \quad 1
\end{align*}
$$

$tz_1^{-1} z_2^{-2} + t^2 z_2^{-1} = (1+tz_1 z_2)tz_1^{-1} z_2^{-2}$
How can we get all factors?

We hope to use a "divide-and-conquer" approach. For instance, consider the $1 + tz_1 z_2$ factor.

**Lemma (Intuitive version)**

We can pair almost all primed symplectic patterns into pairs, so that the weight of each pair is divisible by $1 + tz_1 z_2$, by using a simple algorithm. The patterns that are left have a very restricted and tense structure.

**Idea of proof:**

Most big patterns will have an entry $d$ in the following configuration

| 2nd row  | a | e |
| 3rd row  | b | d | f |
| 4th row  | c | g |

where $a, b, c \gg d \gg e, f, g$. 
Tableaux: Easier Weights

One of our major projects has been to translate our results for Type A into the language of "tableaux", specifically "shifted tableaux".

\[
\begin{array}{ccc}
1 & 1 & 2 & 3 \\
2 & 2 & 3 \\
3
\end{array}
\]

Above is an example of a shifted tableaux of shape \( \lambda = (4, 3, 1) \). Note the entries increasing down the rows and columns.

\[
\Psi \left( \begin{array}{cccc}
1 & 1 & 2 & 3 \\
2 & 2 & 3 \\
3
\end{array} \right) = \begin{array}{ccc}
4 & 3 & 1 \\
3 & 2 \\
2
\end{array}
\]

There's a bijection between semistandard shifted tableaux and GT-patterns. The weight \( m_i \) is extremely simple for tableaux.
Hamel and King

One of our inspirations for trying to reframe our results in the language of tableaux is Hamel and King’s paper "Bijective proofs of shifted tableau and alternating sign matrix identities".

This paper manages to prime semistandard shifted tableaux to prove Tokuyama’s formula. For example, following is a semistandard shifted tableaux of shape $\lambda = (9, 8, 6, 4, 3, 1)$.

$$PST = \begin{array}{cccccccc}
1 & 1 & 1 & 2' & 2 & 2 & 3 & 3 & 5 \\
2 & 2 & 3' & 3 & 4' & 5' & 5' & 5 & 6' \\
3 & 3 & 4' & 4 & 5' & 6 \\
4 & 5' & 5 & 5 \\
5 & 6' & 6 \\
6 \\
\end{array}$$
Conclusion

• Summary of Progress

• Future Directions of Research

• Acknowledgements:
  • Slava Naprienko
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  • SURIM and Stanford

• Main References:
  • Partially Strict Shifted Plane Partitions by Okada
  • Schur Polynomials and the Yang-Baxter Equation by Brubaker, Bump, and Friedberg
  • Simple Proof of Tokuyama’s Formula by Naprienko summarizing work by Bump
  • Symmetric Functions and Hall Polynomials by MacDonald
  • Bijective proofs of shifted tableau and alternating sign matrix identities by Hamel and King
Specializing $t$

After factoring out all of the pairs from the combinatorial sum, the remaining factor (inside the parentheses below) is independent of $t$:

$$\sum T \text{weight}(T) = \prod_{i<j} (z_i + t z_j) \left( \frac{\sum T \text{weight}(T)}{\prod_{i<j} (z_i + t z_j)} \right)$$

Setting $t = -1$ in this factor gives us a formula for the Schur polynomial.
Properties of pairing algorithm

Lemma (Well-defined)

*The pairing algorithm always gives an output.*

Proof.

Not having an output is equivalent to the following configuration:

```
  a
(b + 1)'
```

Now \( a = b \) or \( b + 1 \). But this is a contradiction. \( \square \)
Properties of Combinatorial Pairing Algorithm

Lemma (Pairing)

*The pairing algorithm is an involution, i.e. it is its own inverse.*

**Proof.**

Whether the pairing algorithm can change a particular entry in the second row depends only on the entries in the third row.

**Example:** We cannot change $3 \to 2$:

\[
\begin{array}{ccc|ccc}
3 & 1 & 0 & 3 & 1 & 0 \\
3' & 1' & ; & 3' & 1' & \\
3' & 2 & \\
\end{array}
\]

We can change $3' \to 2$:

\[
\begin{array}{ccc|ccc}
3 & 1 & 0 & 3 & 1 & 0 \\
3' & 1' & ; & 3' & 1' & \\
2' & 1 & \\
\end{array}
\]