COLORED BOSONIC MODELS AND MATRIX COEFFICIENTS
(FIGURES ONLY)

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Abstract. We develop the theory of colored bosonic models (initiated by Borodin and Wheeler). We will show how a family of such models can be used to represent the values of Iwahori vectors in the “spherical model” of representations of \( \text{GL}_r(F) \), where \( F \) is a nonarchimedean local field. Among our results are a monochrome factorization, which is the realization of the Boltzmann weights by fusion of simpler weights, a local lifting property relating the colored models with uncolored models, and an action of the Iwahori Hecke algebra on the partition functions of a particular family of models by Demazure-Lusztig operators. As an application of the local lifting property we reprove a theorem of Korff evaluating the partition functions of the uncolored models in terms of Hall-Littlewood polynomials. Our results are very closely parallel to the theory of fermionic models representing Iwahori Whittaker functions developed by Brubaker, Bump, Buciumas and Gustafsson, with many striking relationships between the two theories, confirming the philosophy that the spherical and Whittaker models of principal series representations are dual.

Motivation from \( \text{GL}_r(F), \ F \) nonarchimedean local.

Spherical principal series against \( \Pi_2 \to \mathbb{Z} \mapsto \varphi(c^r)_{\pi_2} \)

Two very similar stories

(1) Whittaker functions

(2) "Spherical" matrix coeff.

\[ G = \text{GL}_r(F), \ \ K = \text{GL}_r(\mathbb{O}), \ \ O = \text{integers} \]

Maximal compact

\[ J = \text{Iwahori subgroup} : \{ g \in K \mid g \equiv (g_\pi) \mod J \} \]
\[ [K : J] = (q^{r-1})(q^r - q) \cdots (q^r - q^{r-1}) \]

\[ (\pi, V) \quad V \text{ infinite dim.'} \]

\[ \dim(V^\mathbb{C}) = 1 \quad \dim(V^J) = |W| = r! \]

**Problem 1**: \( V \) has a unique linear \( F_N \).

\[ \lambda_{\text{wht}} : V \rightarrow \mathbb{C} \]

\[ \lambda_{\text{wht}}(\pi(w)v) = \chi(w) \lambda(v) \]

\[ N = (\begin{smallmatrix} 1 & \cdots & 1 \\ \vdots & \ddots & \vdots \\ 1 & \cdots & 1 \end{smallmatrix}) \quad \psi : N \rightarrow \mathbb{C}^\times \text{ is a} \]

"Non-degenerate" char.

**Theorem**: (BBBG) For a basis \( \phi_w \) (\( w \in W \)) of invariant vectors, any \( \text{gen} \)

\[ w \in W \text{ whose partition function is} \]

\[ W_w(g) := \chi(\pi(g)\phi_w) \]

"Colored Fermionic Models"
Problem 2: $\exists!$ Functional

$\lambda_{\text{Sph}} : V_\mathbb{C} \to \mathbb{C}$ \hspace{1cm} $\lambda(\pi(\xi)v) = \lambda(v)$

for $\xi \in \mathbb{K}$.

As evidence these stories are parallel,

BBL L = UCATA
BBF F = FRIEDBERG
BBBF

Whittaker, spherical story lead to reps of affine Weyl group,

$\text{IND}_{\mathbb{W}}^{\mathbb{W}^{\text{aff}}(\mathfrak{so}_n)}$ white case demazure-white ops.
$\text{IND}_{\mathbb{W}}^{\mathbb{W}^{\text{aff}}(\mathfrak{sl}_n)}$ spherical case demazure-lusztig ops.

Theorem (BN) so far proven, should work completely for a basis of $\phi_W$, any $g \in G \subseteq \text{SLM}$ such that $\lambda(\pi_z(g)\phi_W) = z(\text{SLM})$

These models, bosonic colored.
Bosonic Models

These models are related to Hall-Littlewood polynomials (related via "Macdonald Formula" to p-adic story.)

BW models are more general, but the special case we are interested in have special props.
(1) **Monochrome Factorization**

(2) "**Local lifting property**" relating colored models $U_q(\hat{\mathfrak{g}}_{r+1})$ to uncolored "**Korff-like**" $U_q(\hat{\mathfrak{g}}_r)$.

(3) Very strong parallels to Whittaker lattice models.

Korff P-models

Uncolored

Korff P-models

Variants R-models

Uncolored case P-models $\Rightarrow$ HL P-Poly.

R-models $\Rightarrow$ HL R-Poly.

$$\Omega(t) = \sum_{\text{even}} (-1)^{\ell(w)} 2^{\omega(w) + 1} \prod_{\alpha \in \Phi^+} (1 - 2^{-\alpha} t^{-1})$$

$$= \Delta_{\lambda}(t) \text{ if } \lambda \text{ dominant}$$

$$R(\lambda, t) = \Delta(\lambda) \prod_{\alpha \in \Phi^+} (1 - t \alpha^{-1})$$

$\lambda$ dominant

$$P_{\lambda}(z, t) = \frac{1}{\Delta_{\lambda}(t)} P_{\lambda}(z, t)$$

See Macdonald III
Figure 1. The grid with boundary conditions for the uncolored model, $\mathcal{G}_\lambda^P(z)$ or $\mathcal{G}_\lambda^R(z)$, corresponding to the partition $\lambda = (8, 6, 6, 1, 0)$, with $r = 5$. A state of the model will assign spins to the interior model.

$$\lambda = (8, 6, 6, 1, 0)$$

**Theorem (Korff):** $Z_P(\mathcal{G}_\lambda^\text{unc}(z)) = \rho_\lambda(z, t)$

**Variant:** $Z_R(\mathcal{G}_\lambda^\text{unc}(z)) = R_\lambda(z, t)$

**The $R$-models have better locality properties**

**We have a new proof:**

output of $\text{MIE} \Rightarrow \text{PF}$ are symmetric, doesn't seem to determine them. **We have a proof of this using colored models.**
<table>
<thead>
<tr>
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<td>$P$-weights</td>
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Figure 2. Uncolored Boltzmann weights of two types: the $P$-weights (which coincide with those in [13], and the $R$-weights.

Consider horizontal edges have states 
\[ \{+,-,\} \]

Vertical have states \[ \{0,1,2,...,\} \]

"$q$-harmonic oscillator"

"valid name for $U_q(sl_2)$ on $\hat{\Delta}_2$"
Figure 3. The Yang-Baxter equation. The partition functions of the two small 3-vertex systems are the same. Here $a, b, c, d, e, f$ are the fixed boundary spins, and in each case we sum over the possible spins of the three interior edges. We may use either the $P$- or the $R$-weights. **Uncolored case:** Use the weights from Figure 2 and the $R$-matrix from 4. **Colored case:** Use the colored weights obtained by fusion (Figure 9) from the monochrome weights (Figure 8).
Figure 4. The uncolored R-matrix. This works for either the uncolored $R$- or $P$-models.

Remarkable, these $U_q(\widehat{A}_2)$ models are related to $U_q(C_{1|1})$ models by a single modification.

Change first entry to $z_i - tz_j$ and get R-matrix for models whose partition functions are (free-fermionic)

$$\prod_{t \in \mathbb{D}} (1 - t^2 z^{-1}_i) \Delta_{\lambda}(z)$$

"Tokuyama models"

\[ \text{Casselman-Shalika Whittaker Functions.} \]
Figure 5. The R-matrix. This R-matrix is nearly identical to Figure 6 in [3], with one difference: for the first entry (++++), the R-matrix value is $z_i - tz_j$ here, but $z_j - tz_i$ in [3]. This small difference makes a difference in the quantum group: in [3] the quantum group is a Drinfeld twist of $U_q(\widehat{sl}(r|1))$, but this R-matrix is that of $U_q(\widehat{gl}(r + 1))$. (Compare [12], equation (3.5).)
hecke algebra action.

describe bdy conditions & "flags"

flag c

\[
\text{SAME FLAG } c
\]

if no top flag = top flag only one state

so

\[
z(s_{\lambda}, c, c) = (*2) Z_{\lambda}
\]

up

same flags

yang baxter eq'n! \quad c_0 = standard flag,

\[
z(s_{\lambda}, c_0, a_1 w c_0) = Z z(s_{\lambda}, c_0, u c_0)
\]

\[
A_i = \text{simple reflection}
\]

provided \( A_i w > w \).
Figure 6. Auxiliary Yang-Baxter equations. The auxiliary R-matrix labeled $z_i, z_j, \gamma_k$ is from Figure 7. Our convention is that $\gamma_0 = \gamma_k$, which agrees with the R-matrix in Figure 5. Note that we are using the monochrome vertices so $c, f \in \Sigma_{\gamma_0, \gamma_k}^{\text{mon}}$. This Yang-Baxter equation may be proved by direct examination of the possible cases.

\[ \begin{array}{c}
\begin{tikzpicture}
\draw (0,0) node [left] {$a$} to [bend left] (1,1) node [above] {$b$} to [bend left] (2,0) node [right] {$c$};
\draw (1,1) -- (1,2) node [above] {$d$};
\draw (2,0) -- (2,2) node [above] {$e$};
\end{tikzpicture}
\end{array} \]
Colored weights with a monochrome factorization have a signature:
certain weights must be zero.

\[
\begin{array}{cccc}
\text{CARRIES} & \text{CARRIES} \\
\text{ONLY RED} & \text{ONLY GREEN}
\end{array}
\]

No possible way to supply these colors must be zero.

Horizontal edges carry at most one color.

BW 2018 (1.7.2) shows this characteristic if \( \Delta = 0 \) (also BBBS, \( \{1,4,7,10\} \), "metatone").

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**Figure 7.** R-vertices for auxiliary Yang-Baxter equations. These are labeled by the spectral parameters $z_i$, $z_j$ and a color $c$. Except for the first entry, these weights are identical to the weights in [3], Figure 11, where $z_i - tz_j$ in this paper is replaced by $z_j - tz_i$. This seemingly minor difference changes the quantum group from $U_q(\mathfrak{gl}(1,r))$ in [3] to $U_q(\widehat{\mathfrak{gl}}(r+1))$ in this paper.
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### Figure 8

Monochrome vertex type \( z_i, c \). The vertical edge can carry only the color \( c \). Thus the possible states of the horizontal edges are all colors and \( + \). Possible spins of the vertical edges are labeled by integers \( n \), representing \( n \) copies of the boson of color \( c \).

\[
\mathcal{L}_i \phi = (z^\sigma - 1)^{-1}(\phi - z^\sigma \phi - t(\phi - z^\sigma \phi))
\]

This algebraic expression shows how Hecke algebra acts on functions.

We also prove matrix coefficients in spherical model satisfy same recursion.
Figure 9. Fusion. The colors $\gamma_i$ are ordered so that $\gamma_r < \gamma_{r-1} < \cdots < \gamma_1$, so they are arranged in increasing order from left to right. This procedure replaces a sequence of vertices by a single vertex. Here $z_i, \gamma_i$ is the “monochrome” vertex from Figure 8. The single fused vertex $z_i$ replaces the $r$ unfused vertices $z_{i}, \gamma_{i}$. If $b, d \in \Sigma_v^{\text{col}}$ then write $b = \gamma_1^{b_1} \cdots \gamma_r^{b_r} \in \Sigma_v^{\text{col}}$ for $(b_1, \cdots, b_r) \in \prod_{i=1}^{r} \Sigma_v^{\text{mon}}$, and similarly write $d = \gamma_1^{d_1} \cdots \gamma_r^{d_r} \in \Sigma_v^{\text{col}}$. The Boltzmann weight of the fused vertex (call it $v$) is just the partition function of the configuration on the right-hand side of the figure. That is, there will be a unique assignment of spins to the internal edges on the left such that the Boltzmann weights are nonzero, and $\beta(v)$ is the product of the Boltzmann weights of the monochrome vertices, from Figure 8.
Figure 10. The unique state of $\mathcal{S}_{\lambda, c, v}(z)$ for $G = \text{GL}_3$ where $\lambda = (4, 2, 2)$, $c = s_2 c_0 = (R, B, G)$ and $c_0 = (\gamma_1, \gamma_2, \gamma_3) = (R, B, G)$. Top: the fused model. Bottom: the corresponding state in the unfused model.
Figure 11. Proof of Proposition 4.4. Top: the system $\mathcal{S}_{\lambda,c,d}(s,z)$ with the R-matrix attached. Bottom: after using the Yang-Baxter equation.
Figure 12. R-values needed in the proof of Proposition 4.4. These may be read off from Figure 5 bearing in mind that $c_i > c_{i+1}$.
References


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