Stochastic symplectic ice: a stochastic vertex model with U-turn boundary

Chenyang Zhong

Stanford University

February 24, 2021

Outline

Table of contents

Motivation

Symplectic ice[1] (Cartan type C): vertex model with U-turn boundary, partition function representing the product of a deformation of Weyl's denominator and an irreducible character of the symplectic group $Sp(2n, \mathbb{C})$

Stochastic six-vertex model[2] (Cartan type A): vertex model with stochastic weights, probabilistically interpreted as an interacting particle system

Question: Is there a stochastic vertex model that is related to other Cartan types and allows probabilistic interpretation?

In this work, we present a stochastic vertex model with U-turn boundary, with connections to Cartan type C

Paper: Stochastic symplectic ice, arxiv:2102.00660

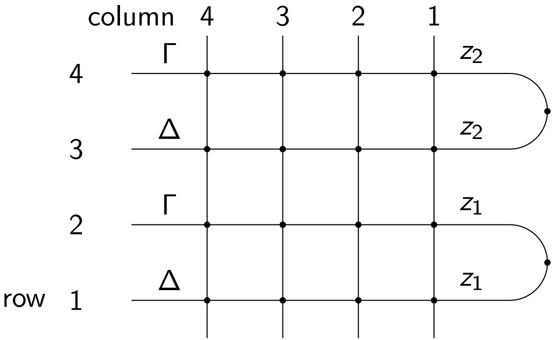
[1] Dmitriy Ivanov. "Symplectic ice". In: *Multiple Dirichlet series, L-functions and automorphic forms.* Springer, 2012, pp. 205–222.

[2] Alexei Borodin, Ivan Corwin, Vadim Gorin, et al. "Stochastic six-vertex model". In: *Duke Mathematical Journal* 165.3 (2016), pp. 563–624.

The model: rectangular lattice

- Rectangular grid with $2n$ rows and L columns
- Columns are numbered from right to left, and rows are numbered from bottom to top
- Every edge carries a "$+$" or "$-$" spin
- Two types of vertices called stochastic Γ ice (even-numbered rows) and stochastic Δ ice (odd-numbered rows). The Boltzmann weights for them depend on a parameter called the "spectral parameter"
- The ith row of stochastic Γ and Δ ice: connected by a "cap" on the right, both with spectral parameter z_i

The model: rectangular lattice



The model: boundary data

Boundary set-up depends on

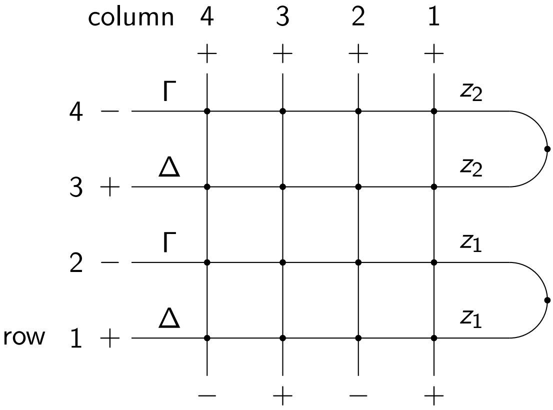
- A partition $(\lambda_1, \lambda_2, \cdots, \lambda_N)$ with $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_N$

Boundary set-up:

- Top: "$+$" spin
- Bottom: the column labeled $\lambda_i + N + 1 - i$ is assigned the "$-$" spin, and the rest columns are assigned the "$+$" spin
- Left: every row of stochastic Γ ice is assigned the "$-$" spin, and every row of stochastic Δ ice is assigned the "$+$" spin

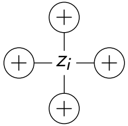
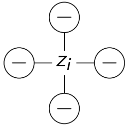
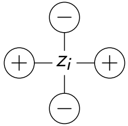
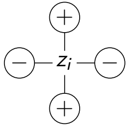
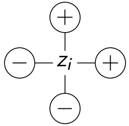
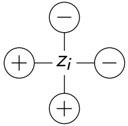
The model: boundary data

Model configuration when $n = 2, \lambda = (2,1), L = 4$:



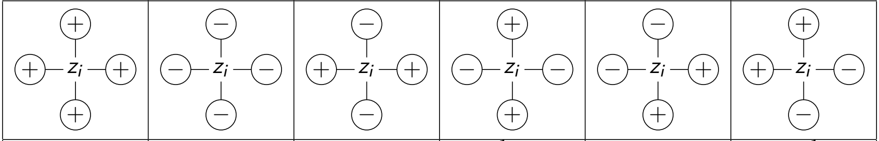
Boltzmann weights for stochastic Γ ice

- Input: left & top edges
- Output: right & bottom edges
- Weights are stochastic (moving rightward): $a_1 = a_2 = 1$, $b_1 + c_2 = 1$, $b_2 + c_1 = 1$

a_1	a_2	b_1	b_2	c_1	c_2
					
1	1	z_i	qz_i	$1 - qz_i$	$1 - z_i$

Boltzmann weights for stochastic Δ ice

- Input: right & top edges
- Output: left & bottom edges
- Weights are stochastic (moving leftward): $a_1 = a_2 = 1$, $b_1 + d_1 = 1$, $b_2 + d_2 = 1$
- $z_i' = q + 1 - \frac{1}{z_i}$
- Choice of the dependence of z_i' on z_i: to ensure integrability (so that the "caduceus relation" holds)

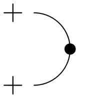
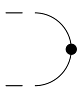


a_1	a_2	b_1	b_2	d_1	d_2
1	1	z_i'	$\frac{1}{q} z_i'$	$1 - z_i'$	$1 - \frac{1}{q} z_i'$

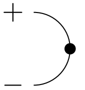
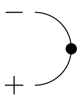
Boltzmann weights for caps

Two choices of cap weights, both stochastic (moving downward)

- Reflecting: corresponding partition function denoted by $Z(\mathcal{S}_{n,L,\lambda,z})$

Cap		
Boltzmann weight	1	1

- Absorbing-and-emitting: corresponding partition function denoted by $Z(\mathcal{T}_{n,L,\lambda,z})$

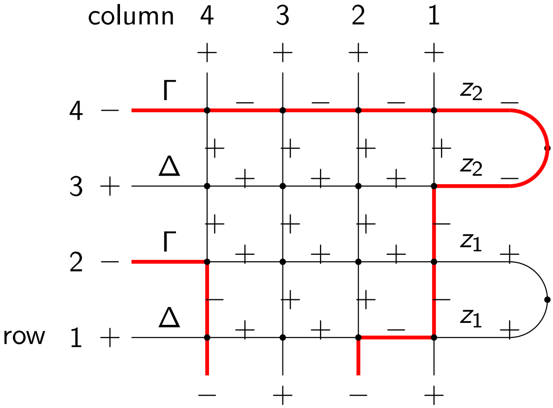
Cap		
Boltzmann weight	1	1

Path interpretation

- Admissible states of the model can be equivalently interpreted as collection of paths (consisting of "$-$" spins)
- The paths move rightward/downward on stochastic Γ ice, and move leftward/downward on stochastic Δ ice

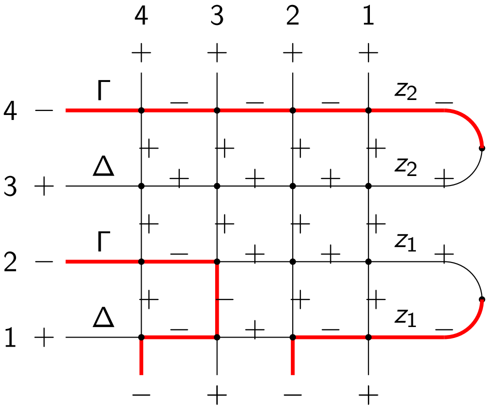
Path interpretation: reflecting case

Reflecting case: when a path meets the cap, it bends to the left



Path interpretation: absorbing-and-emitting case

Absorbing-and-emitting case: when a path meets the cap, it is absorbed; if no path meets the cap in a row of stochastic Γ ice, a new path enters from the cap in the row just below it



Stochastic dynamics

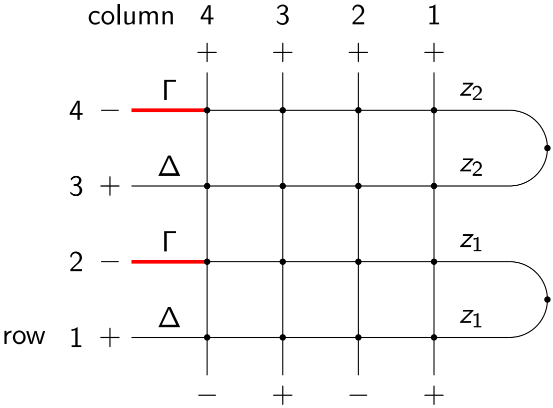
When the condition

$$\max\{0, \frac{1}{q+1}\} \leq z_i \leq \min\{\frac{1}{q}, 1\}, \text{ for every } i \in \{1, 2, \cdots, n\}$$

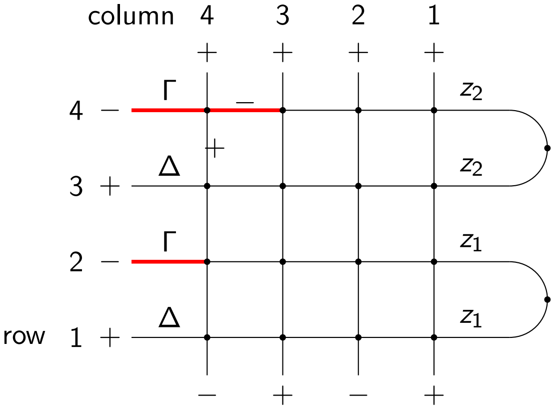
is satisfied, all weights involved are non-negative.

The model can be probabilistically interpreted as stochastic dynamics

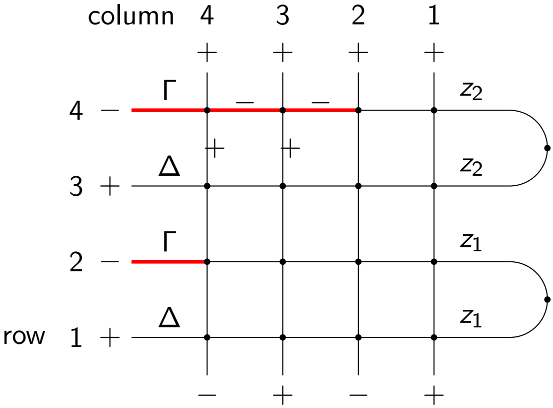
Stochastic dynamics



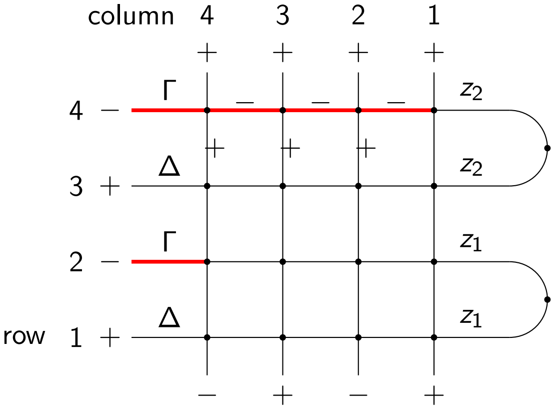
Stochastic dynamics



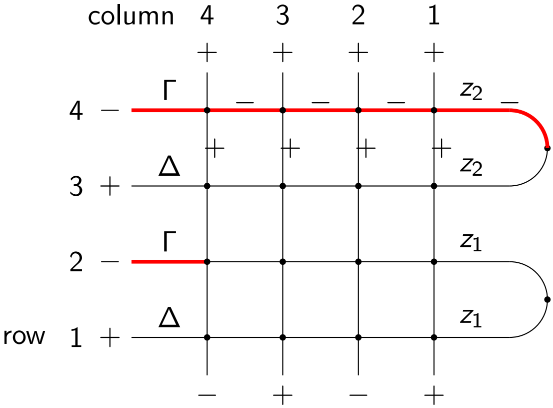
Stochastic dynamics



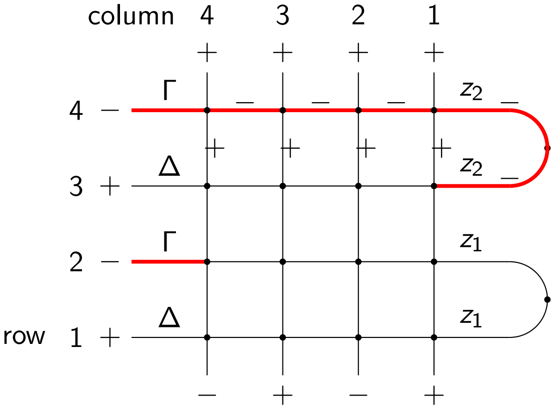
Stochastic dynamics



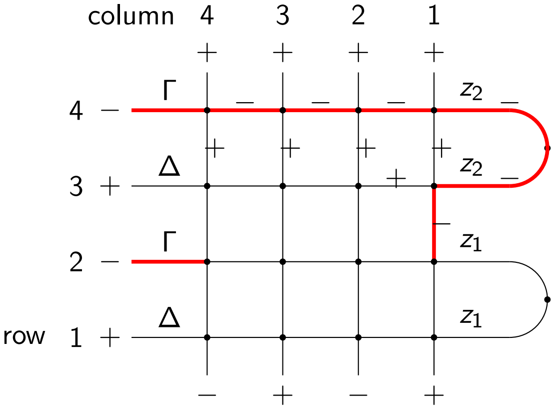
Stochastic dynamics



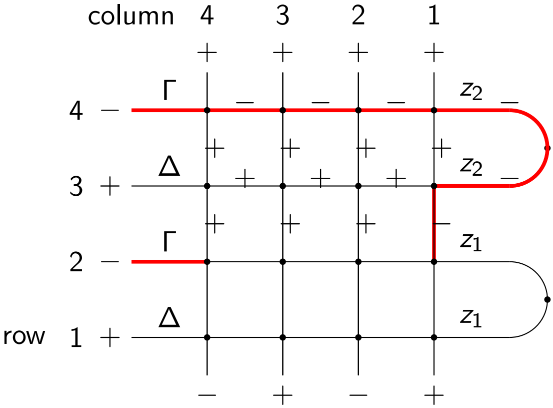
Stochastic dynamics



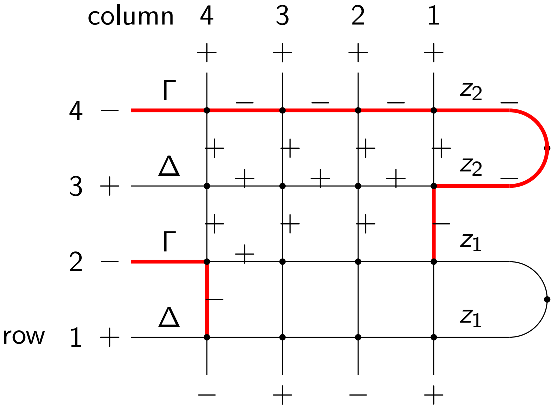
Stochastic dynamics



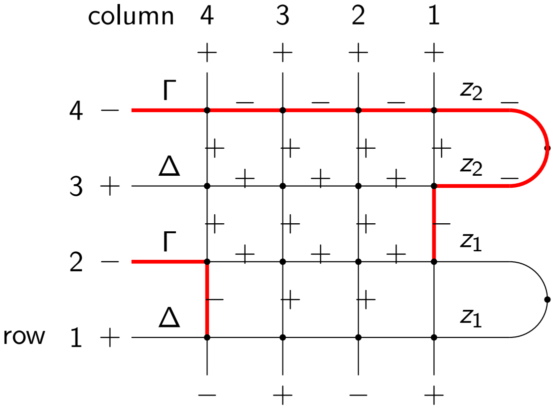
Stochastic dynamics



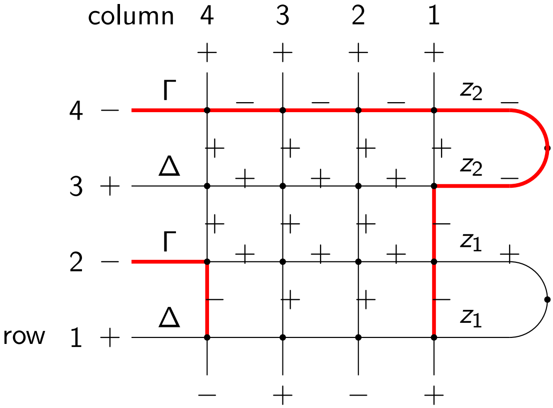
Stochastic dynamics



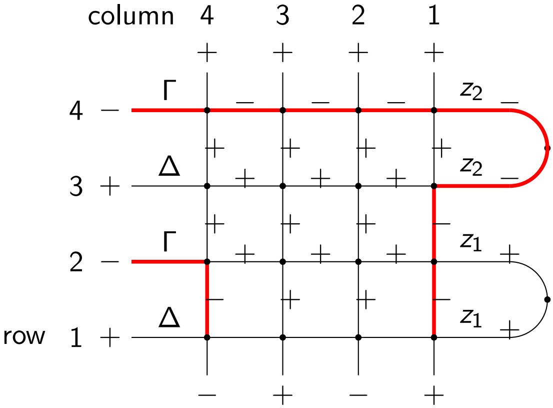
Stochastic dynamics



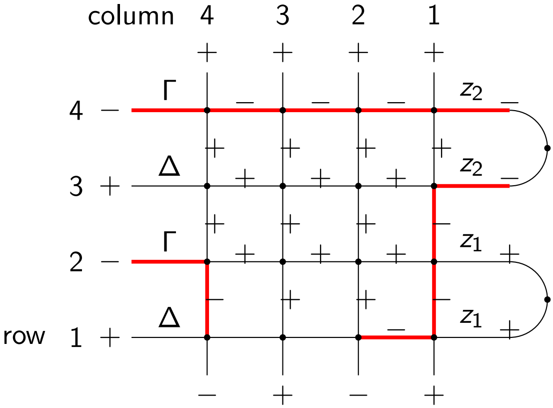
Stochastic dynamics



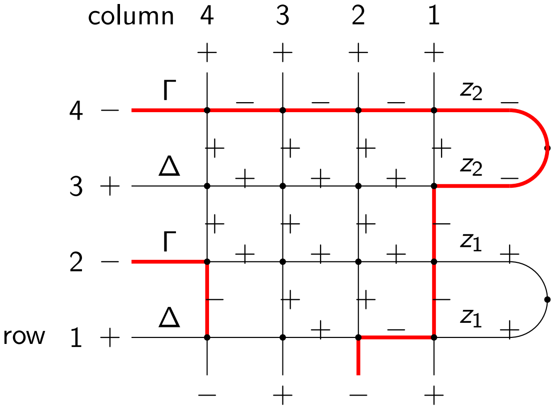
Stochastic dynamics



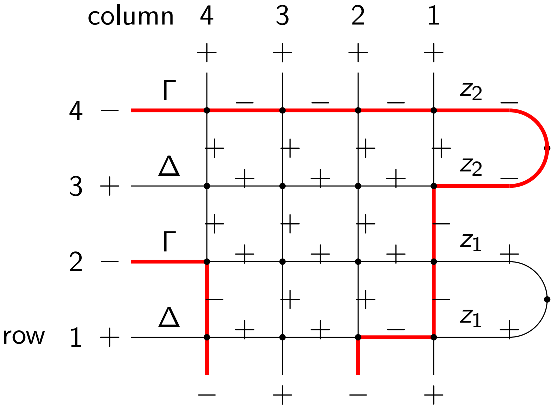
Stochastic dynamics



Stochastic dynamics



Stochastic dynamics

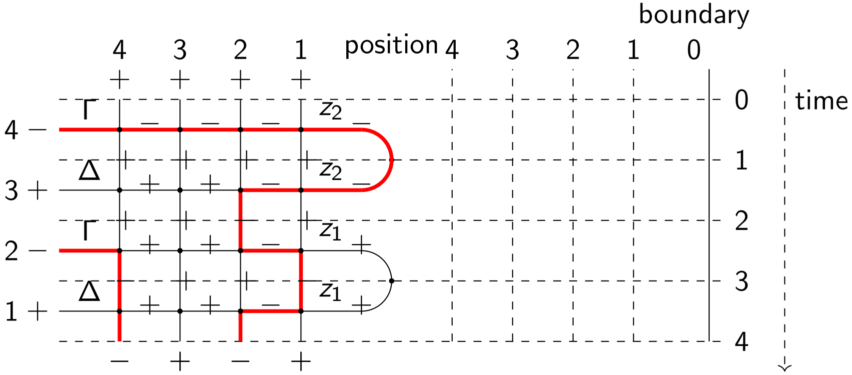


Interacting particle system

- $t = 0, 1, \cdots, 2n$
- View "$-$" spins between the $(2n - t)$th row and the $(2n - t + 1)$th row as particles at time t
- Particles enter from the left
- Particles jump to the right on each row of stochastic Γ ice, and jump to the left on each row of stochastic Δ ice

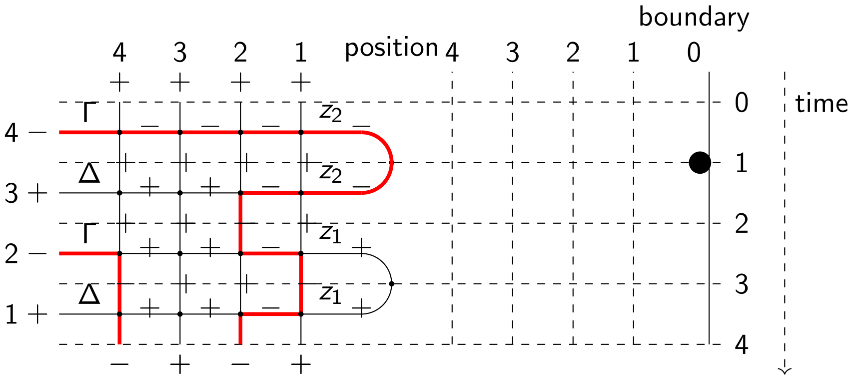
Interacting particle system: the reflecting case

For the reflecting case, when a particle hits the right boundary, it is reflected



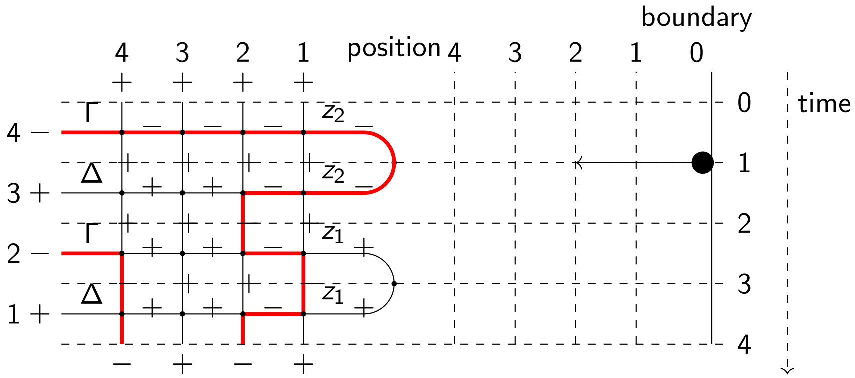
Interacting particle system: the reflecting case

For the reflecting case, when a particle hits the right boundary, it is reflected



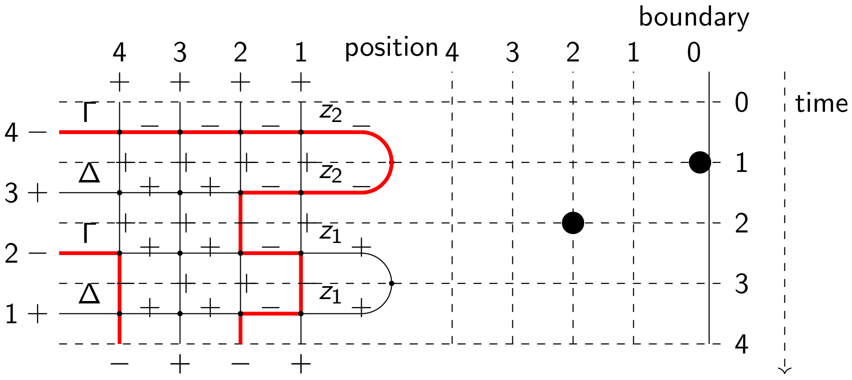
Interacting particle system: the reflecting case

For the reflecting case, when a particle hits the right boundary, it is reflected



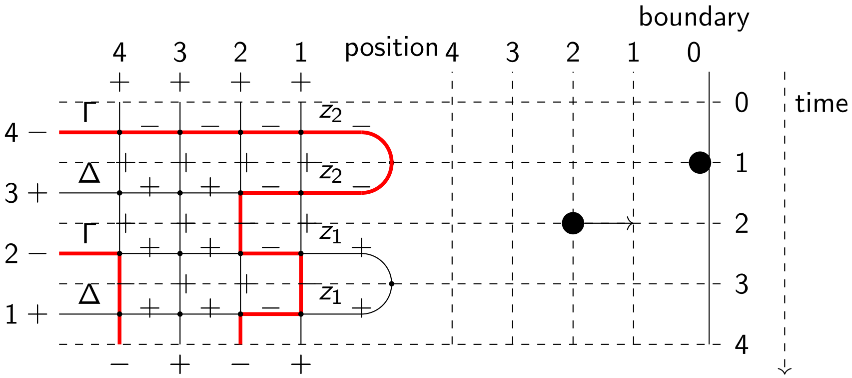
Interacting particle system: the reflecting case

For the reflecting case, when a particle hits the right boundary, it is reflected



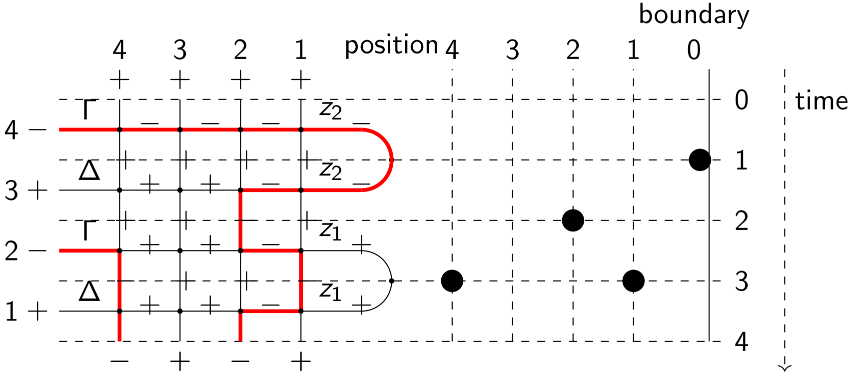
Interacting particle system: the reflecting case

For the reflecting case, when a particle hits the right boundary, it is reflected



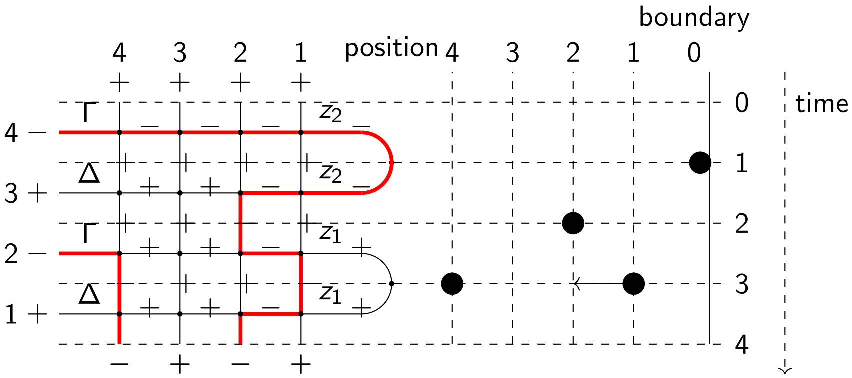
Interacting particle system: the reflecting case

For the reflecting case, when a particle hits the right boundary, it is reflected



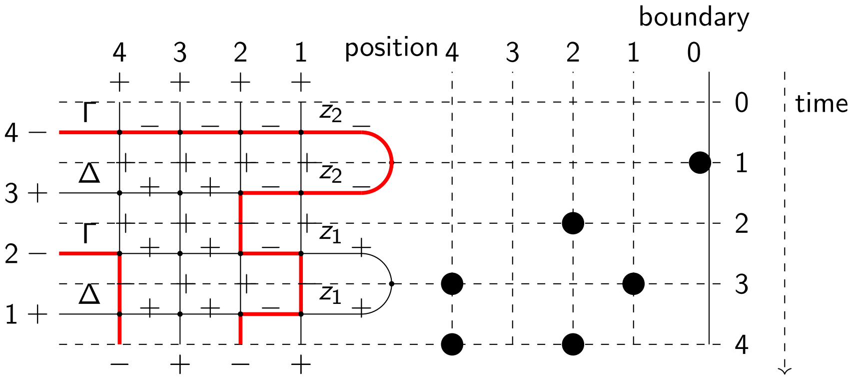
Interacting particle system: the reflecting case

For the reflecting case, when a particle hits the right boundary, it is reflected



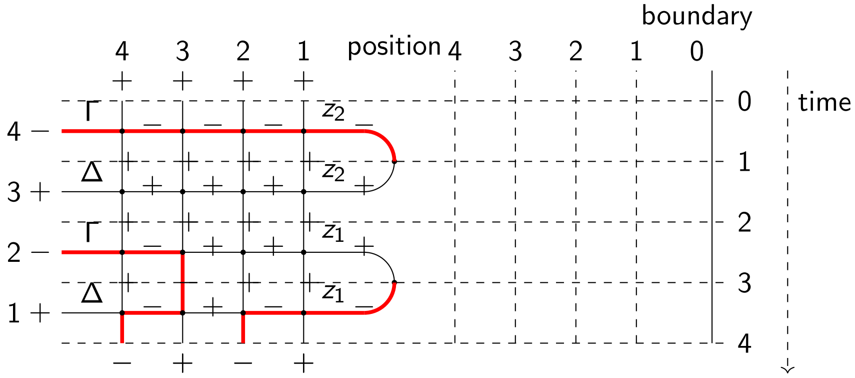
Interacting particle system: the reflecting case

For the reflecting case, when a particle hits the right boundary, it is reflected



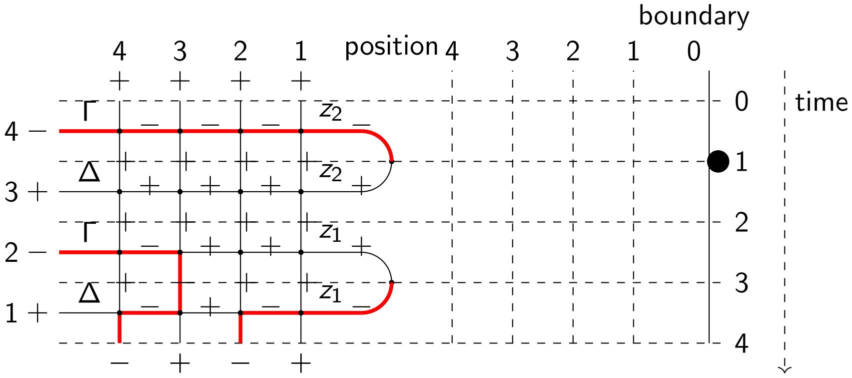
Interacting particle system: the absorbing-and-emitting case

For the absorbing-and-emitting case, when a particle hits the right boundary, it is absorbed; if no particle hits the right boundary, a new particle is emitted from the right boundary



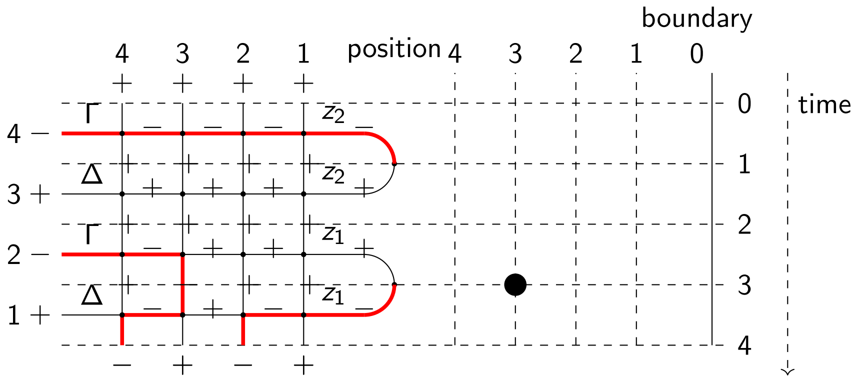
Interacting particle system: the absorbing-and-emitting case

For the absorbing-and-emitting case, when a particle hits the right boundary, it is absorbed; if no particle hits the right boundary, a new particle is emitted from the right boundary



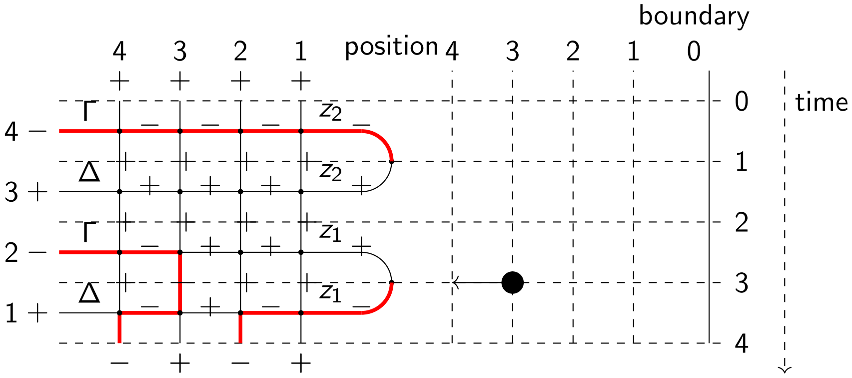
Interacting particle system: the absorbing-and-emitting case

For the absorbing-and-emitting case, when a particle hits the right boundary, it is absorbed; if no particle hits the right boundary, a new particle is emitted from the right boundary



Interacting particle system: the absorbing-and-emitting case

For the absorbing-and-emitting case, when a particle hits the right boundary, it is absorbed; if no particle hits the right boundary, a new particle is emitted from the right boundary



Interacting particle system: the absorbing-and-emitting case

For the absorbing-and-emitting case, when a particle hits the right boundary, it is absorbed; if no particle hits the right boundary, a new particle is emitted from the right boundary

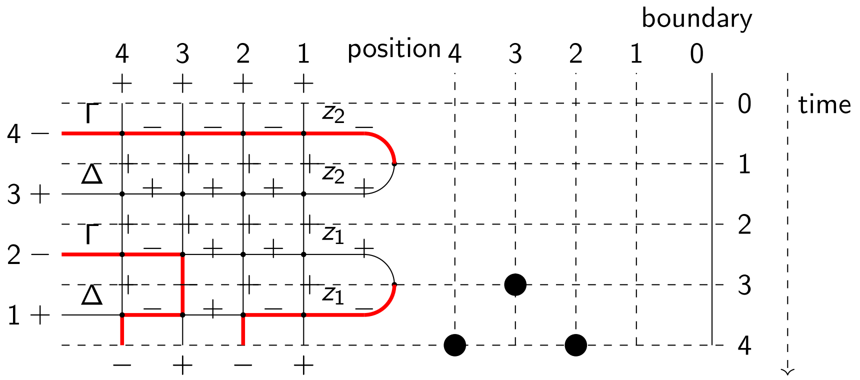


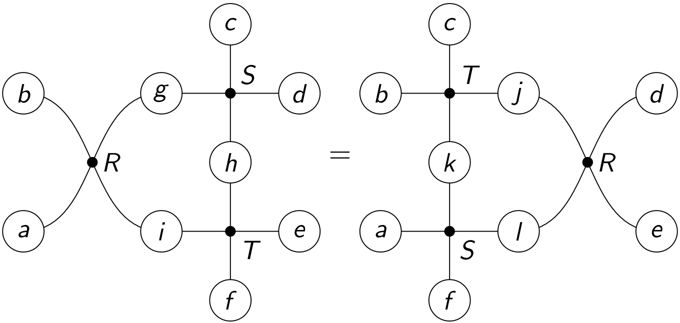
Table of contents

The Yang-Baxter equations

- Stochastic Γ and Δ ice satisfy four sets of Yang-Baxter equations
- The corresponding R-matrices are called stochastic $\Gamma - \Gamma$, $\Gamma - \Delta$, $\Delta - \Gamma$, $\Delta - \Delta$ ice
- R-matrix depends on two spectral parameters z_i and z_j

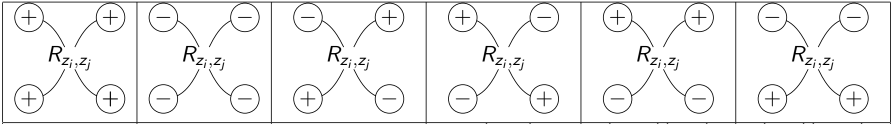
The Yang-Baxter equations

- $X, Y \in \{\Gamma, \Delta\}$
- S: stochastic X ice with spectral parameter z_i
- T: stochastic Y ice with spectral parameter z_j
- R: stochastic $X - Y$ ice with spectral parameters z_i, z_j



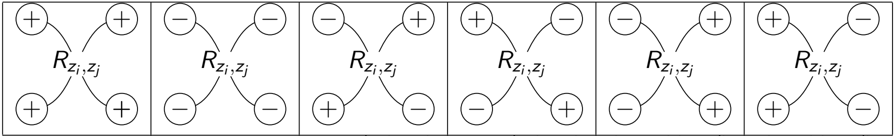
Boltzmann weights for R-matrices

Stochastic $\Gamma - \Gamma$ ice with spectral parameters z_i and z_j



1	1	$\frac{z_i-z_j}{1-(q+1)z_j+qz_iz_j}$	$\frac{q(z_i-z_j)}{1-(q+1)z_j+qz_iz_j}$	$\frac{(1-qz_i)(1-z_j)}{1-(q+1)z_j+qz_iz_j}$	$\frac{(1-z_i)(1-qz_j)}{1-(q+1)z_j+qz_iz_j}$

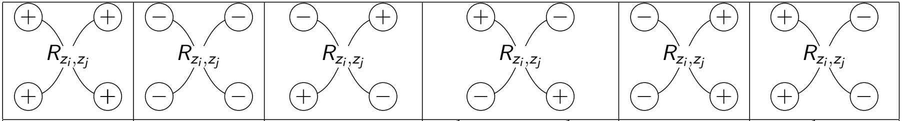
Stochastic $\Gamma - \Delta$ ice with spectral parameters z_i and z_j



1	1	$\frac{qz_i+z_j'-(1+q)}{z_iz_j'-1}$	$\frac{qz_i+z_j'-(1+q)}{q(z_iz_j'-1)}$	$\frac{(1-qz_i)(1-z_j')}{z_iz_j'-1}$	$\frac{(1-z_i)(q-z_j')}{q(z_iz_j'-1)}$

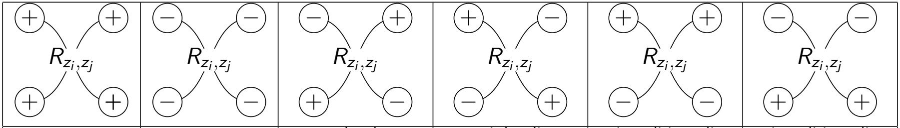
Boltzmann weights for R-matrices

Stochastic $\Delta - \Gamma$ ice with spectral parameters z_i and z_j



1	1	$\frac{z_i'+qz_j-(q+1)z_i'z_j}{1-z_i'z_j}$	$\frac{q^{-1}z_i'+z_j-(1+q^{-1})z_i'z_j}{1-z_i'z_j}$	$\frac{(1-z_i')(1-qz_j)}{1-z_i'z_j}$	$\frac{(1-q^{-1}z_i')(1-z_j)}{1-z_i'z_j}$

Stochastic $\Delta - \Delta$ ice with spectral parameters z_i and z_j



1	1	$\frac{z_j'-z_i'}{q-(q+1)z_i'+z_i'z_j'}$	$\frac{q(z_j'-z_i')}{q-(q+1)z_i'+z_i'z_j'}$	$\frac{(1-z_i')(q-z_j')}{q-(q+1)z_i'+z_i'z_j'}$	$\frac{(1-z_j')(q-z_i')}{q-(q+1)z_i'+z_i'z_j'}$

Functional equations for the partition function

Using the Yang-Baxter equations, functional equations satisfied by the partition function under two types of transformations can be derived

- Permutation of $z_1, \cdots, z_n$
- Interchanging $z_i \leftrightarrow z_i'^{-1}$, $1 \leq i \leq n$

Functional equations for the partition function

Theorem (Z.,2021)

Let

$$D_1(n, L, z) = \prod_{i=1}^{n} z_i^L \prod_{i=1}^{n} (1 - (q+1)z_i + qz_i z_i'^{-1})$$

and

$$D_2(n, L, z) = \prod_{i=1}^{n} z_i^L.$$

Then $\frac{Z(\mathcal{S}_{n,L,\lambda,z})}{D_1(n,L,z)}$ *and* $\frac{Z(\mathcal{T}_{n,L,\lambda,z})}{D_2(n,L,z)}$ *are invariant under any permutation of* $z_1, \cdots, z_n$ *and any interchange* $z_i \leftrightarrow z_i'^{-1}$.

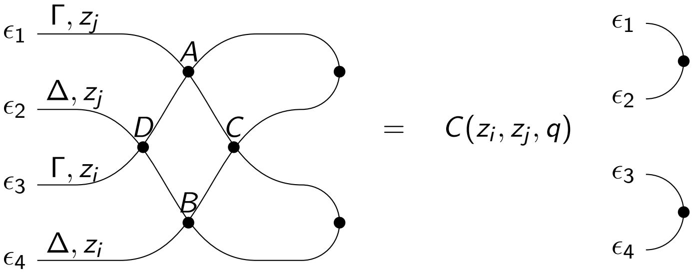
Functional equations for the partition function

The derivation is based on the Yang-Baxter equations and two additional relations

- Caduceus relation
- Fish relation

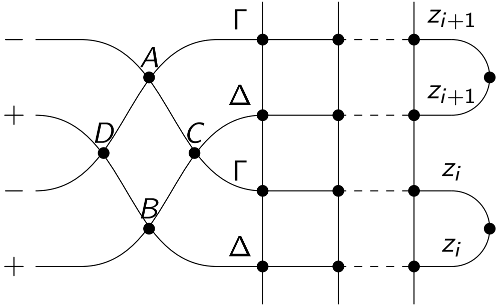
Caduceus relation

- A, B, C, D: stochastic $\Gamma - \Gamma$, $\Delta - \Delta$, $\Delta - \Gamma$, $\Gamma - \Delta$ ice
- Spectral parameters: z_i and z_j
- $C(z_i, z_j, q) = \frac{(qz_iz_j-1)(1-(q+1)(z_i+z_j)+(q^2+q+1)z_iz_j)}{q(z_i+z_j-(q+1)z_iz_j)^2}$



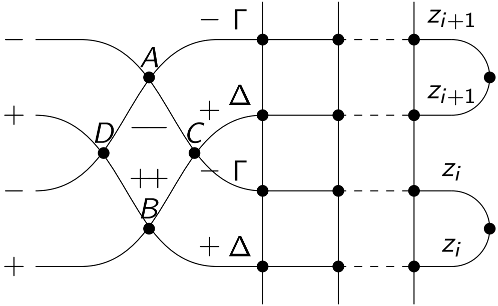
Interchanging z_i and z_{i+1}

Attach the caduceus braid to the left



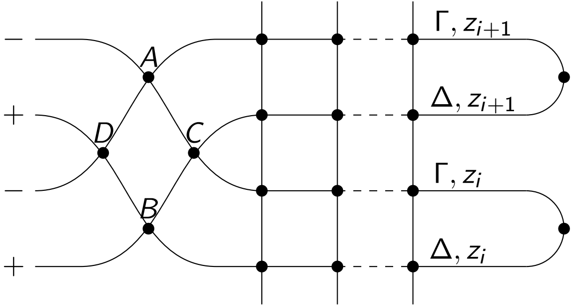
Interchanging z_i and z_{i+1}

Attach the caduceus braid to the left



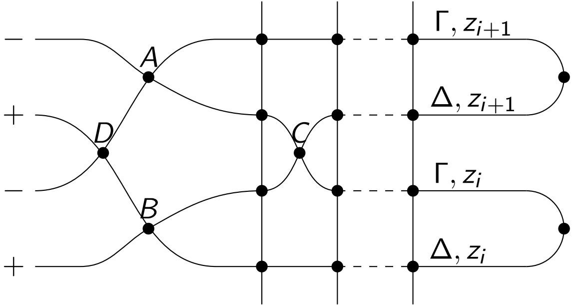
Interchanging z_i and z_{i+1}

Move the braid to the right (in the order of C, A, B, D) using the four sets of Yang-Baxter equations



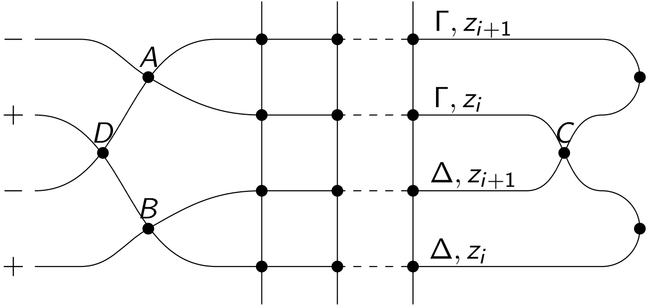
Interchanging z_i and z_{i+1}

Move the braid to the right (in the order of C, A, B, D) using the four sets of Yang-Baxter equations



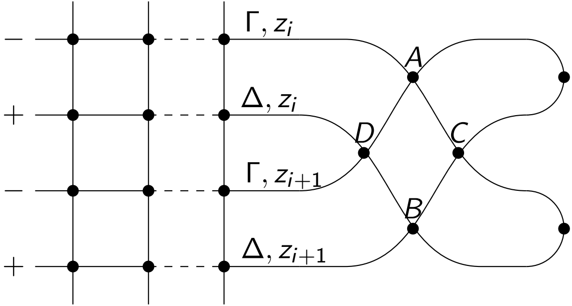
Interchanging z_i and z_{i+1}

Move the braid to the right (in the order of C, A, B, D) using the four sets of Yang-Baxter equations



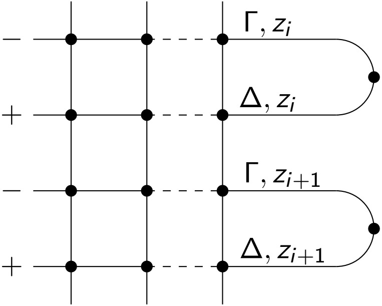
Interchanging z_i and z_{i+1}

Move the braid to the right (in the order of C, A, B, D) using the four sets of Yang-Baxter equations



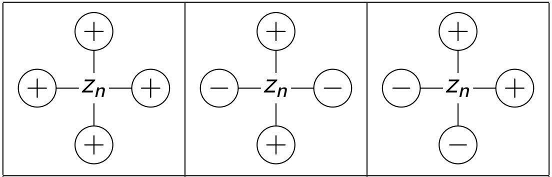
Interchanging z_i and z_{i+1}

By the caduceus relation, the previous partition function is equal to $C(z_{i+1}, z_i, q)$ times the following partition function



Interchanging z_n and $\frac{1}{z'_n}$

Observation: only "+" spin can appear on the top boundary
Therefore, only the following three states are involved in the top row:

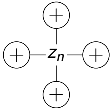
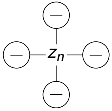
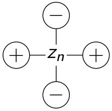
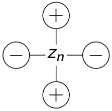
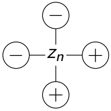
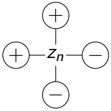


1	qz_n	$1 - qz_n$

Interchanging z_n and $\frac{1}{z_n'}$

We make the following changes

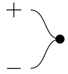
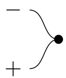
- Interchange "+" and "−" spins in the top row
- Change the Boltzmann weights of the top row to

a_1	a_2	b_1	b_2	d_1	d_2
					
1	1	$\frac{1}{z_n}$	$\frac{1}{qz_n}$	$\frac{1}{z_n} - 1$	$\frac{1}{qz_n} - 1$

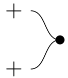
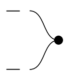
Interchanging z_n and $\frac{1}{z_n'}$

We make the following changes

- For the reflecting case, change the Boltzmann weights of the top cap to

New cap		
Boltzmann weight	1	1

- For the absorbing-and-emitting case, change the Boltzmann weights of the top cap to

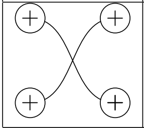
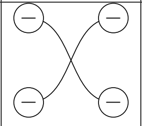
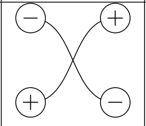
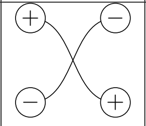
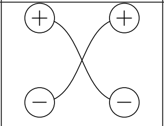
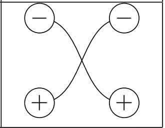
New cap		
Boltzmann weight	1	1

Interchanging z_n and $\frac{1}{z_n'}$

- The Boltzmann weights for each vertex in the top row changes by the factor $\frac{1}{qz_n}$
- The partition function changes by the factor $\frac{1}{(qz_n)^L}$

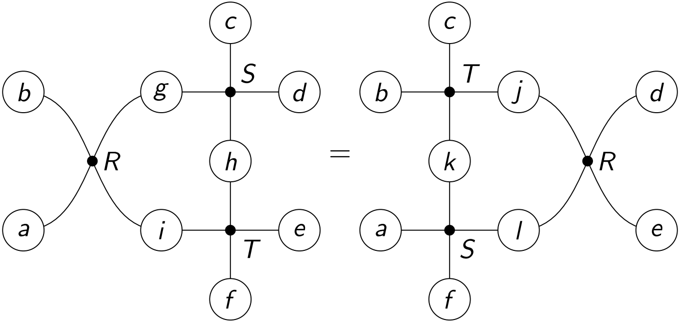
New R-matrix

Introduce the following R-matrix

a_1	a_2	b_1	b_2	c_1	c_2
					
1	1	$\frac{z_n z_n' - 1}{q z_n + z_n' - (q+1)}$	$\frac{q(z_n z_n' - 1)}{q z_n + z_n' - (q+1)}$	$-\frac{(z_n - 1)(q - z_n')}{q z_n + z_n' - (q+1)}$	$-\frac{(q z_n - 1)(1 - z_n')}{q z_n + z_n' - (q+1)}$

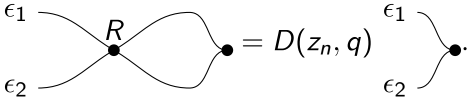
New R-matrix: Yang-Baxter equation

- S: vertex on the top row for the changed system
- T: vertex on the second top row
- R: new R-matrix



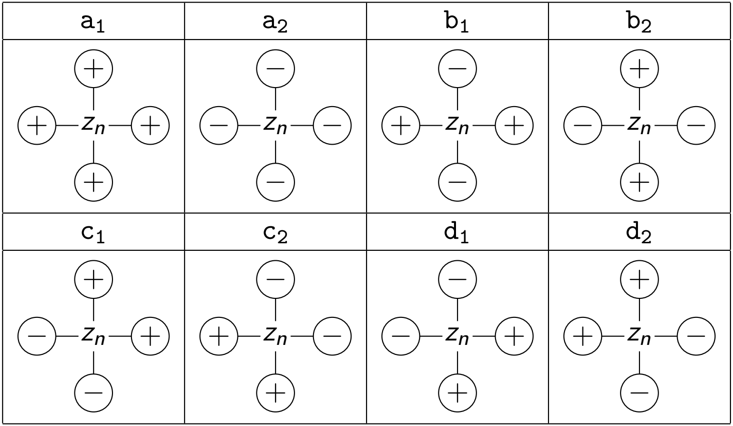
New R-matrix: fish relation

- For the reflecting case, $D(z_n, q) = -\frac{1-(q+1)z_n+qz_nz_n'^{-1}}{1-(q+1)z_n'^{-1}+qz_nz_n'^{-1}}$
- For the absorbing-and-emitting case, $D(z_n, q) = 1$



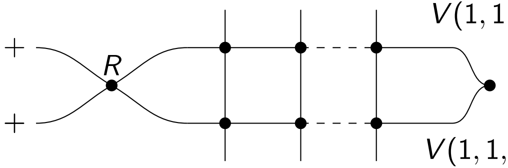
Interchanging z_n and $\frac{1}{z_n'}$: reflecting case

$V(a_1, a_2, b_1, b_2, c_1, c_2, d_1, d_2)$: vertex with Boltzmann weights $a_1, a_2, b_1, b_2, c_2, c_2, d_1, d_2$.



Interchanging z_n and $\frac{1}{z'_n}$: reflecting case

Attach the new R-matrix to the left of the top two rows

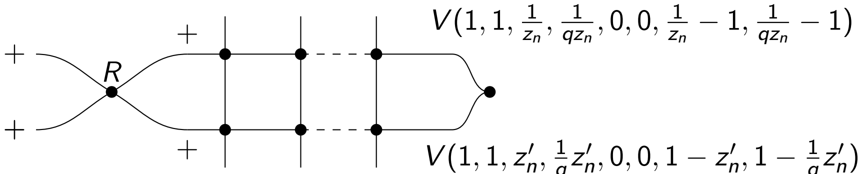


$V(1, 1, \frac{1}{z_n}, \frac{1}{qz_n}, 0, 0, \frac{1}{z_n} - 1, \frac{1}{qz_n} - 1)$

$V(1, 1, z'_n, \frac{1}{q}z'_n, 0, 0, 1 - z'_n, 1 - \frac{1}{q}z'_n)$

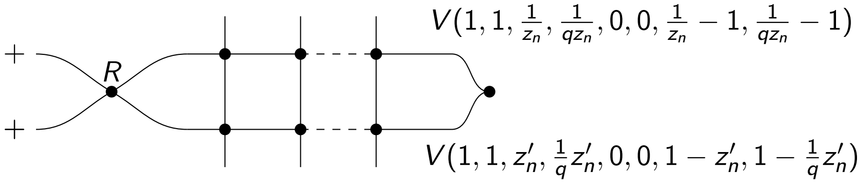
Interchanging z_n and $\frac{1}{z_n'}$: reflecting case

Attach the new R-matrix to the left of the top two rows



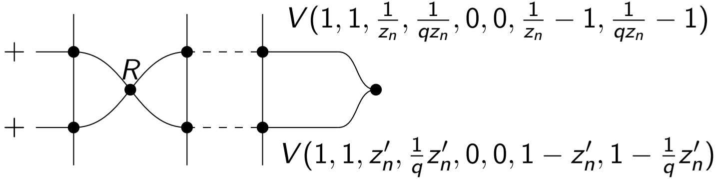
Interchanging z_n and $\frac{1}{z_n'}$: reflecting case

Push the new R-matrix to the right using the Yang-Baxter equation



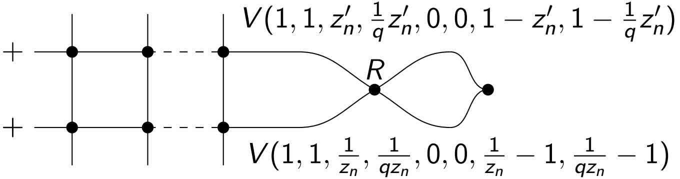
Interchanging z_n and $\frac{1}{z_n'}$: reflecting case

Push the new R-matrix to the right using the Yang-Baxter equation



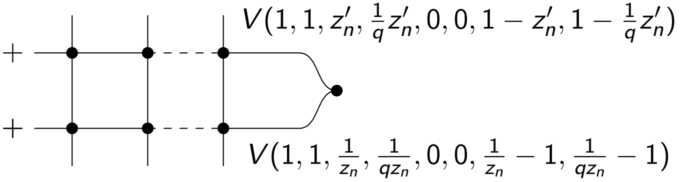
Interchanging z_n and $\frac{1}{z_n'}$: reflecting case

Push the new R-matrix to the right using the Yang-Baxter equation



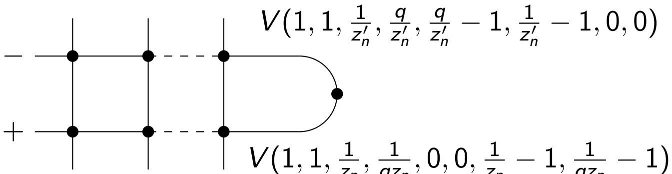
Interchanging z_n and $\frac{1}{z_n'}$: reflecting case

By the fish relation, the previous partition function is equal to $D(z_n, q)$ times the following



Interchanging z_n and $\frac{1}{z'_n}$: reflecting case

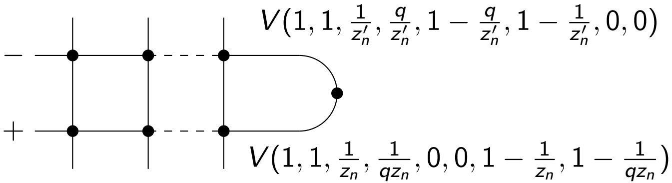
- Interchange "+" and "−" spins in the top row
- Change the Boltzmann weights of the top row
- Change the Boltzmann weights of the cap back to the original one
- The partition function changes by a factor of $(\frac{q}{z'_n})^L$



Interchanging z_n and $\frac{1}{z'_n}$: reflecting case

The total number of c_1, c_2, d_1, d_2 patterns in the top tow rows is an odd number

Hence the previous partition function is equal to -1 times the following



Interchanging z_n and $\frac{1}{z'_n}$: reflecting case

The total number of c_1, c_2, d_1, d_2 patterns in the top tow rows is an odd number

Hence the previous partition function is equal to -1 times the following

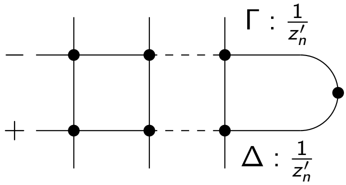


Table of contents

Signed color

- Introduce $2n$ colors $[\pm n] = \{\overline{n}, \cdots, \overline{1}, 1, \cdots, n\}$
- Every edge carries either a "$+$" spin or one of the $2n$ colors
- View the "$+$" spin as color 0
- Order the colors by $\overline{n} < \cdots < \overline{1} < 0 < 1 < \cdots < n$

Hyperoctahedral group B_n

- Presentation of B_n:

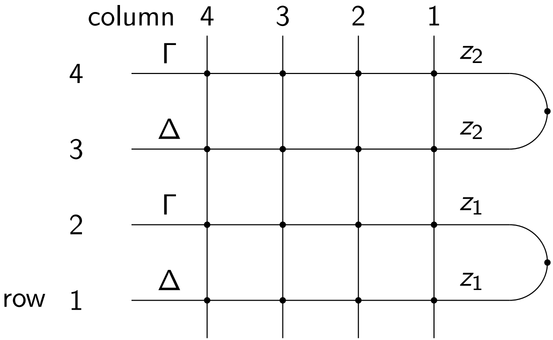
$$B_n = \langle s_1, \cdots, s_n | s_i^2 = 1, 1 \le i \le n; (s_i s_{i+1})^3 = 1, 1 \le i \le n-2; \\ (s_{n-1} s_n)^4 = 1; (s_i s_j)^2 = 1, 1 \le i < j \le n, |i-j| > 1\rangle.$$

- B_n is the Weyl group for type C_n root system
- An element $\sigma \in B_n$ can be viewed as the permutation of $[\pm n]$ such that $\sigma(\bar{i}) = \overline{\sigma(i)}$

The colored model: rectangular lattice

- Rectangular grid with $2n$ rows and L columns
- Columns are numbered from right to left, and rows are numbered from bottom to top
- Two types of vertices: called colored stochastic Γ ice (even-numbered rows) and colored stochastic Δ ice (odd-numbered rows)
- The ith row of colored stochastic Γ and Δ ice: connected by a "cap" on the right, both with spectral parameter z_i

The colored model: rectangular lattice



The colored model: boundary data

Boundary set-up depends on the following

- A partition $(\lambda_1, \lambda_2, \cdots, \lambda_n)$ with $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$
- Two signed permutations $\sigma, \tau \in B_n$

Boundary set-up:

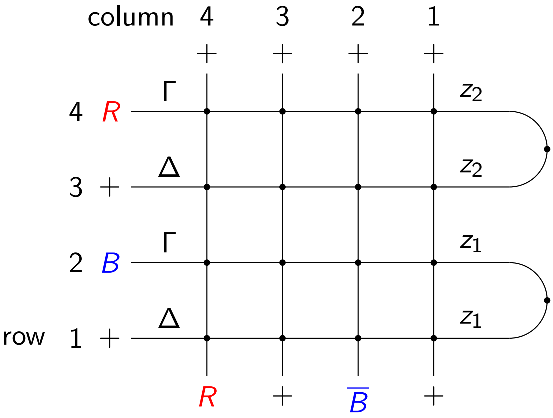
- Top: "$+$" spin
- Bottom: the column labeled $\lambda_i + n + 1 - i$ is assigned the color $\tau(i)$, and the rest columns are assigned the "$+$" spin
- Left: the ith row of colored stochastic Γ ice is assigned the color $\sigma(i)$, and every row of colored stochastic Δ ice is assigned the "$+$" spin

The partition function is denoted by $Z(\mathcal{S}_{n,L,\lambda,\sigma,\tau,z})$.

The colored model: boundary data

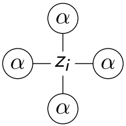
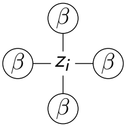
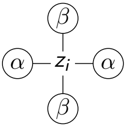
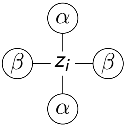
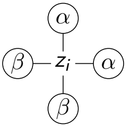
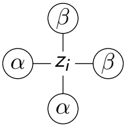
Model configuration when $n = 2, \lambda = (2,1), L = 4$:

$$\overline{R} < \overline{B} < B < R$$



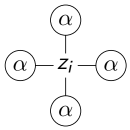
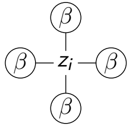
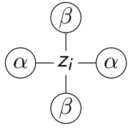
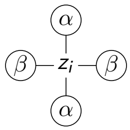
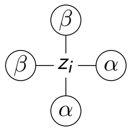
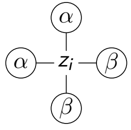
Boltzmann weights for colored stochastic Γ ice

- $\alpha < \beta$
- The weights are stochastic (moving rightward)

a_1/a_2	a_1/a_2	b_1	b_2	c_1	c_2
					
1	1	z_i	qz_i	$1 - qz_i$	$1 - z_i$

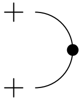
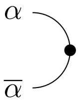
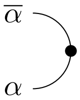
Boltzmann weights for colored stochastic Δ ice

- $\alpha < \beta$
- The weights are stochastic (moving leftward)
- $z_i' = q + 1 - \frac{1}{z_i}$

a_1/a_2	a_1/a_2	b_1	b_2	d_1	d_2
					
1	1	z_i'	$\frac{1}{q} z_i'$	$1 - z_i'$	$1 - \frac{1}{q} z_i'$

Boltzmann weights for the caps

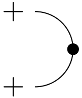
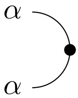
- $\alpha \in \{1, 2, \cdots, n\}$
- The weights are stochastic (moving downward)

Cap			
Boltzmann weight	1	1	1

Alternative model

There is an alternative integrable colored model (which satisfies the Yang-Baxter equations and the reflection equation–to be introduced later) where only positive colors $\{1, \cdots, n\}$ are involved. See the paper[3] for details.

For that model, the Boltzmann weights for the caps are given by ($\alpha \in \{1, \cdots, n\}$)

Cap		
Boltzmann weight	1	1

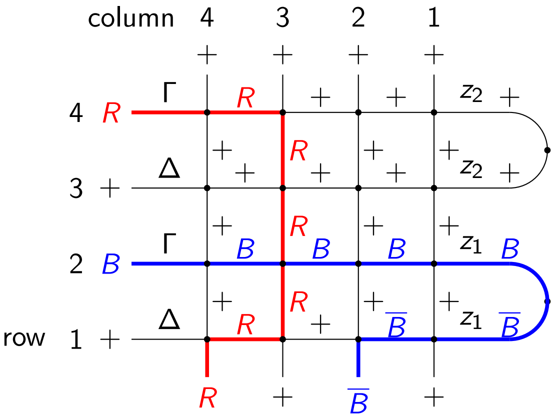
[3] Chenyang Zhong. "Stochastic symplectic ice". In: *arXiv preprint arXiv:2102.00660* (2021).

Colored path interpretation

- Admissible states of the model can be equivalently interpreted as collection of colored paths
- The paths move rightward/downward on colored stochastic Γ ice, and move leftward/downward on colored stochastic Δ ice
- When a path meets the cap, it bends to the left with its color changed to the opposite

Colored path interpretation

$$\overline{R} < \overline{B} < B < R$$



Colored stochastic dynamics

Similar to the uncolored case, when the condition

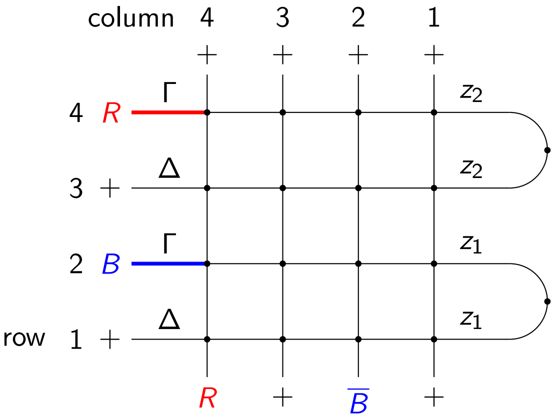
$$\max\{0, \frac{1}{q+1}\} \leq z_i \leq \min\{\frac{1}{q}, 1\}, \text{ for every } i \in \{1, 2, \cdots, n\}$$

is satisfied, all weights involved are non-negative.

The model can be probabilistically interpreted as stochastic dynamics with colors

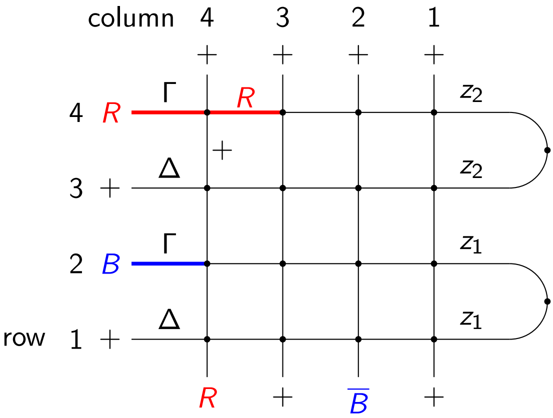
Colored stochastic dynamics

$$\overline{R} < \overline{B} < B < R$$



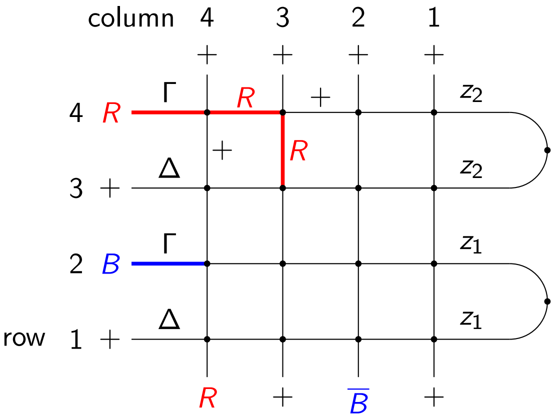
Colored stochastic dynamics

$$\overline{R} < \overline{B} < B < R$$



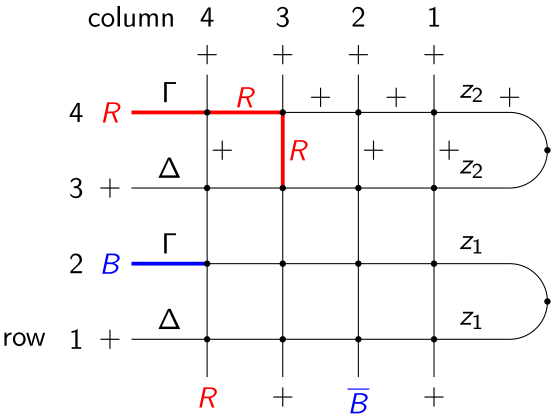
Colored stochastic dynamics

$$\overline{R} < \overline{B} < B < R$$



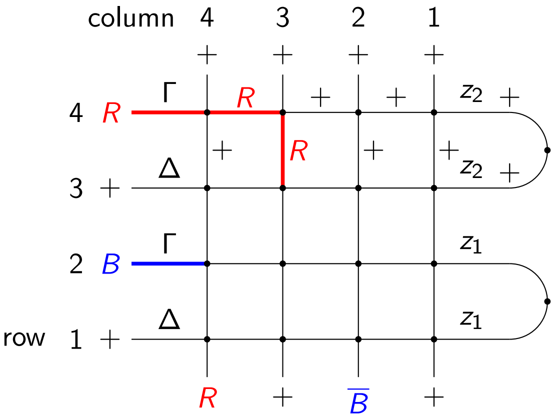
Colored stochastic dynamics

$$\overline{R} < \overline{B} < B < R$$



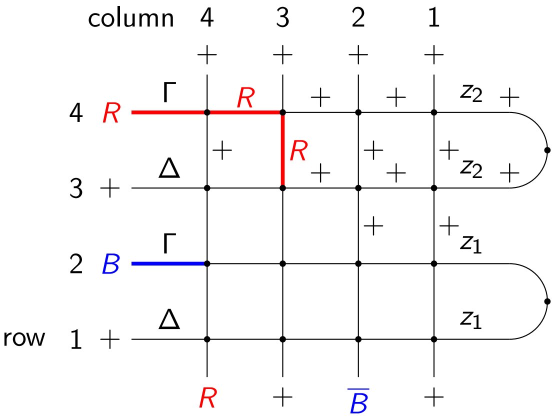
Colored stochastic dynamics

$$\overline{R} < \overline{B} < B < R$$



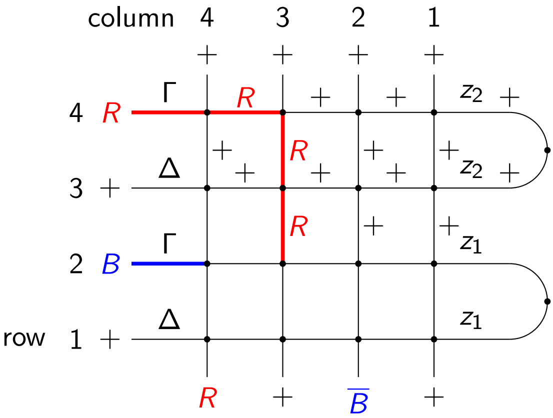
Colored stochastic dynamics

$$\overline{R} < \overline{B} < B < R$$



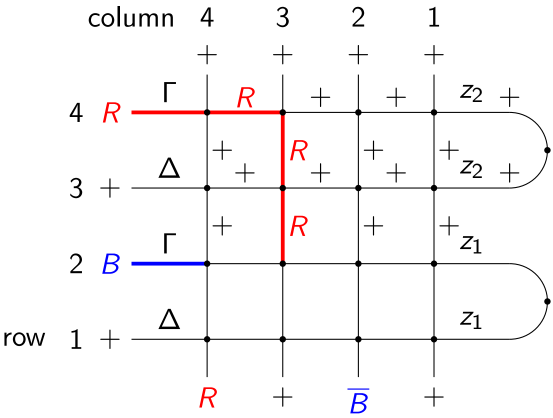
Colored stochastic dynamics

$$\overline{R} < \overline{B} < B < R$$



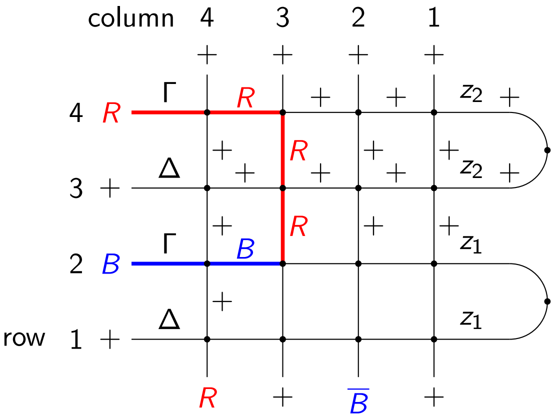
Colored stochastic dynamics

$$\overline{R} < \overline{B} < B < R$$



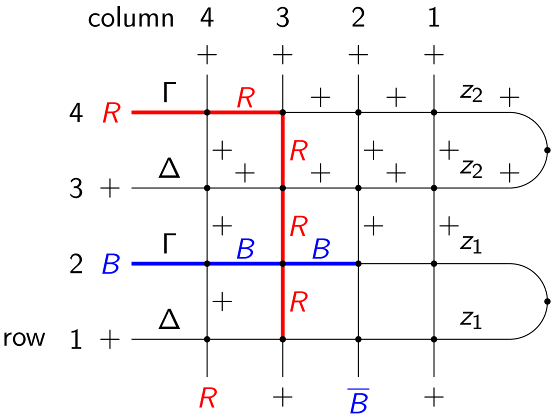
Colored stochastic dynamics

$\overline{R} < \overline{B} < B < R$



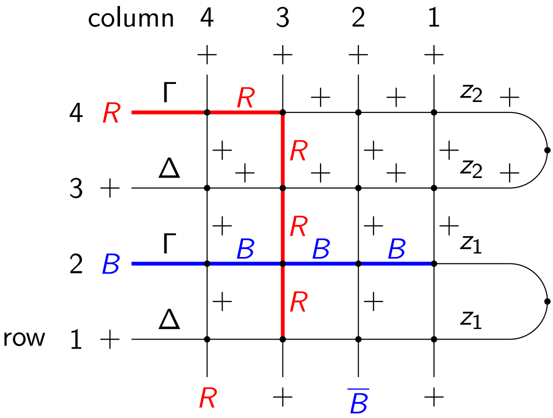
Colored stochastic dynamics

$$\overline{R} < \overline{B} < B < R$$



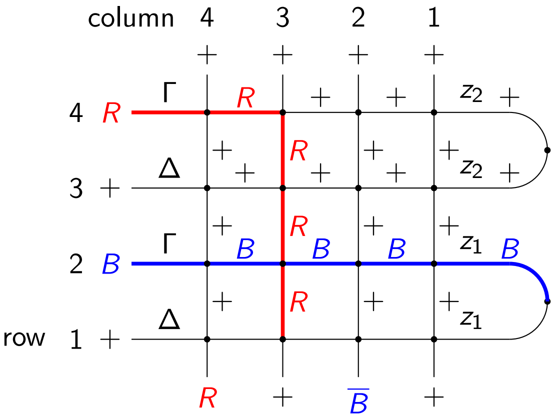
Colored stochastic dynamics

$$\overline{R} < \overline{B} < B < R$$



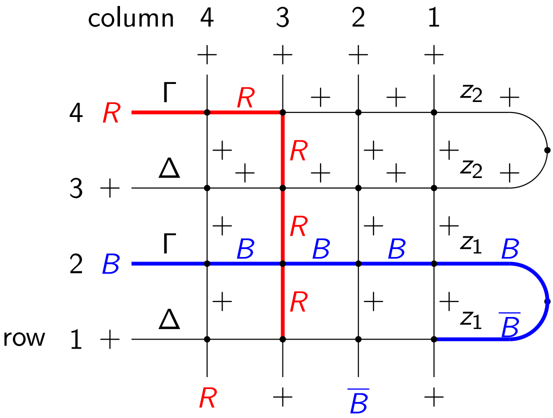
Colored stochastic dynamics

$$\overline{R} < \overline{B} < B < R$$



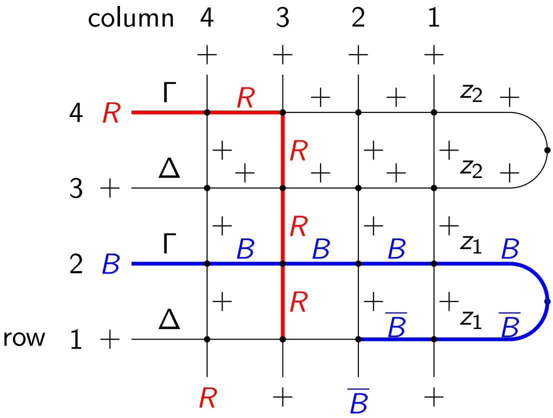
Colored stochastic dynamics

$\overline{R} < \overline{B} < B < R$



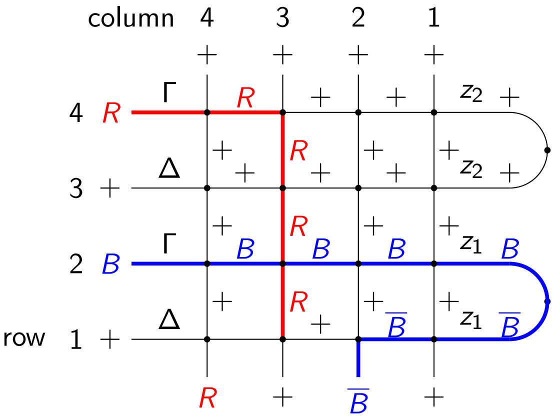
Colored stochastic dynamics

$$\overline{R} < \overline{B} < B < R$$



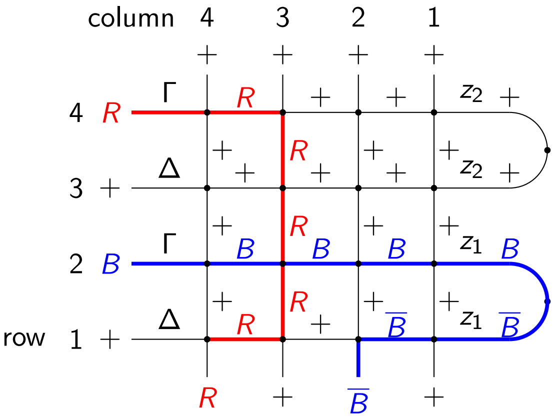
Colored stochastic dynamics

$$\overline{R} < \overline{B} < B < R$$



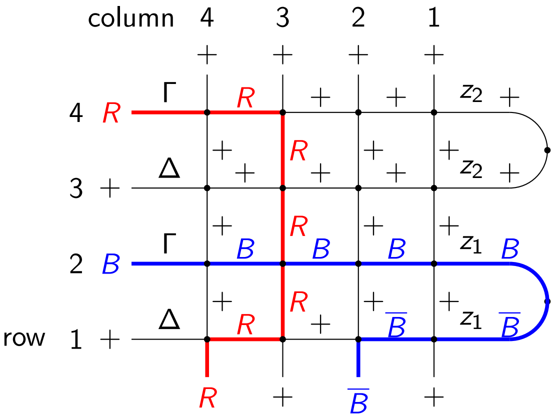
Colored stochastic dynamics

$$\overline{R} < \overline{B} < B < R$$



Colored stochastic dynamics

$$\overline{R} < \overline{B} < B < R$$

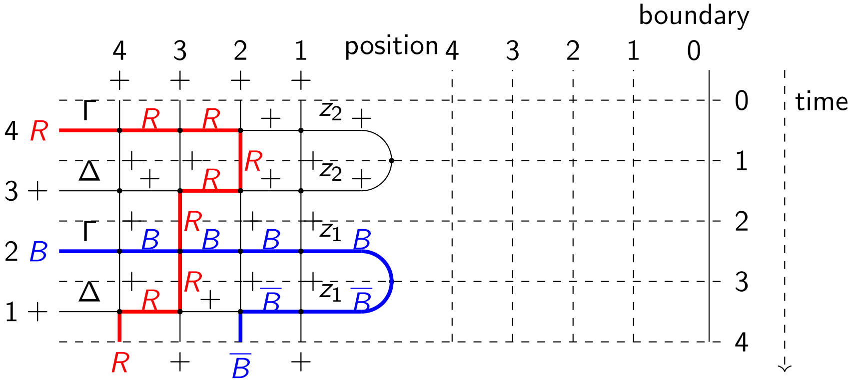


Interacting particle system

- $t = 0, 1, \cdots, 2n$
- View edges carrying a color between the $(2n - t)$th row and the $(2n - t + 1)$th row as particles at time t
- Each particle carries a color
- Particles enter from the left
- Particles jump to the right on each row of colored stochastic Γ ice, and jump to the left on each row of colored stochastic Δ ice
- When a particle hits the right boundary, it is reflected, with its color changed to the opposite

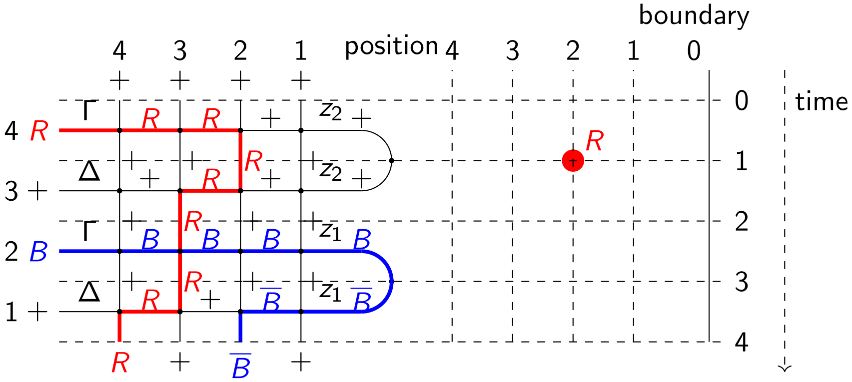
Interacting particle system

$$\overline{R} < \overline{B} < B < R$$



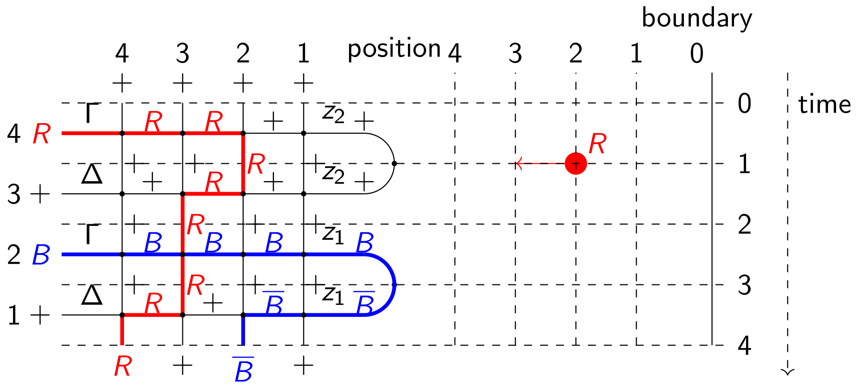
Interacting particle system

$$\overline{R} < \overline{B} < B < R$$



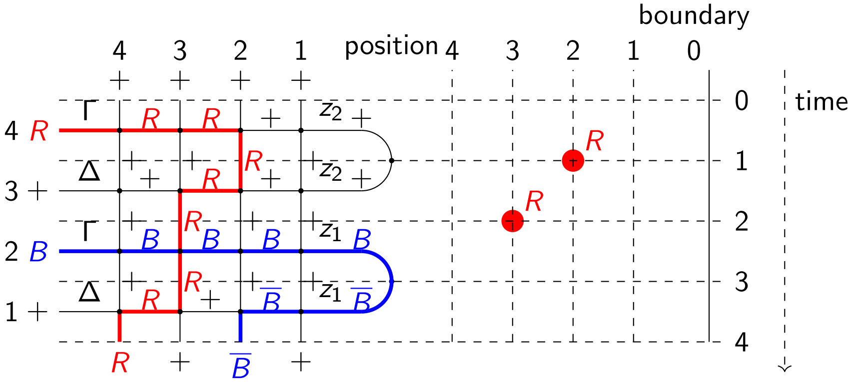
Interacting particle system

$$\overline{R} < \overline{B} < B < R$$



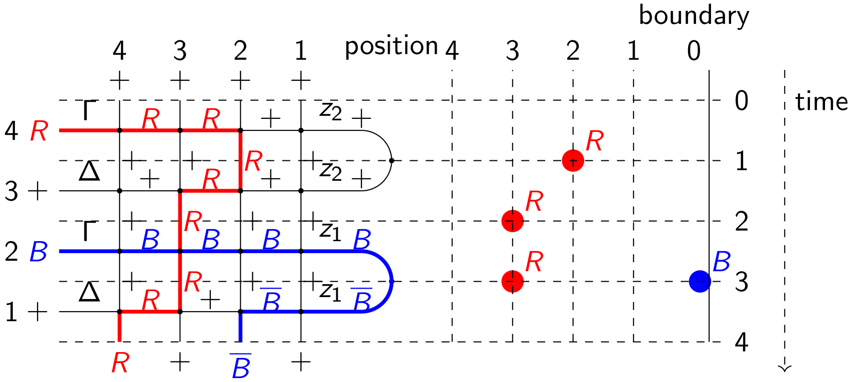
Interacting particle system

$$\overline{R} < \overline{B} < B < R$$



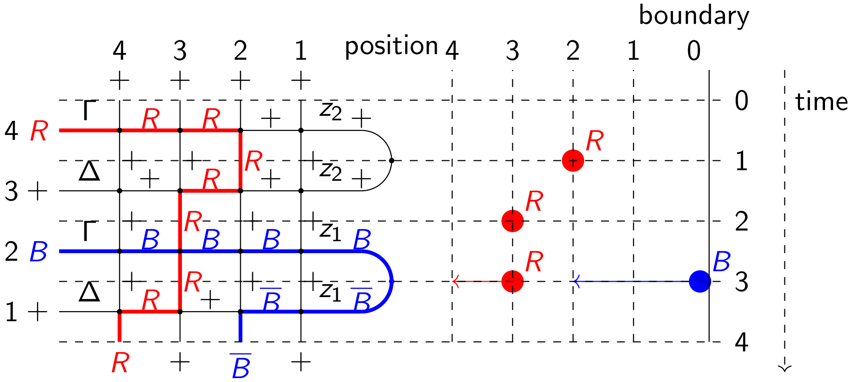
Interacting particle system

$$\overline{R} < \overline{B} < B < R$$



Interacting particle system

$$\overline{R} < \overline{B} < B < R$$



Interacting particle system

$$\overline{R} < \overline{B} < B < R$$

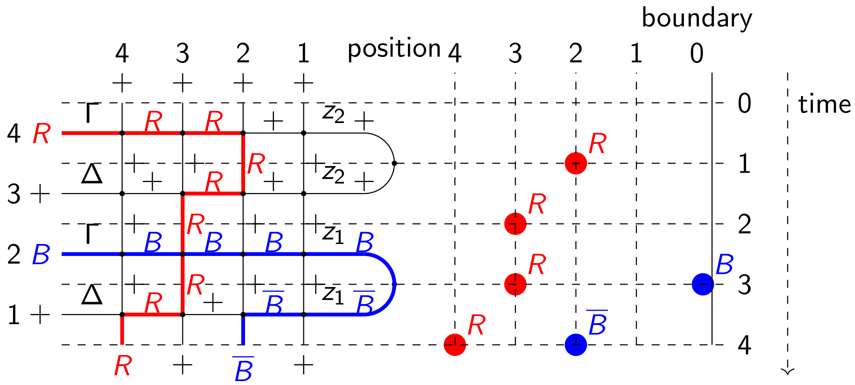


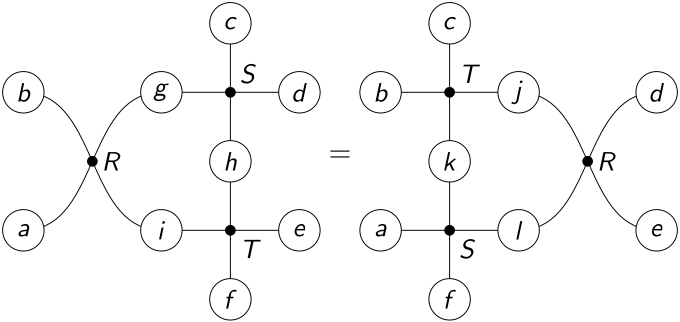
Table of contents

The Yang-Baxter equations

- Colored stochastic Γ and Δ ice satisfy **three** sets of Yang-Baxter equations
- The corresponding R-matrices are called colored stochastic $\Gamma - \Gamma$, $\Delta - \Gamma$, $\Delta - \Delta$ ice
- R-matrix depends on two spectral parameters z_i and z_j

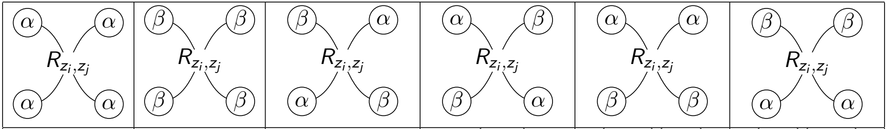
The Yang-Baxter equations

- $(X, Y) = (\Gamma, \Gamma), (\Delta, \Gamma), (\Delta, \Delta)$
- S: colored stochastic X ice with spectral parameter z_i
- T: colored stochastic Y ice with spectral parameter z_j
- R: colored stochastic $X - Y$ ice with spectral parameters z_i, z_j



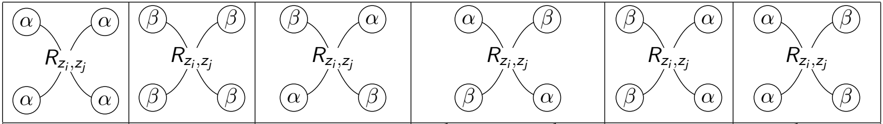
Boltzmann weights for R-matrices

Colored stochastic $\Gamma - \Gamma$ ice with spectral parameters z_i and z_j $(\alpha < \beta)$



1	1	$\frac{z_i - z_j}{1-(q+1)z_j+qz_iz_j}$	$\frac{q(z_i - z_j)}{1-(q+1)z_j+qz_iz_j}$	$\frac{(1-qz_i)(1-z_j)}{1-(q+1)z_j+qz_iz_j}$	$\frac{(1-z_i)(1-qz_j)}{1-(q+1)z_j+qz_iz_j}$

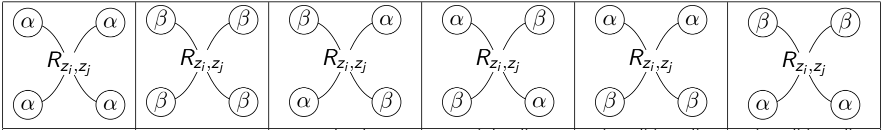
Colored stochastic $\Delta - \Gamma$ ice with spectral parameters z_i and z_j $(\alpha < \beta)$



1	1	$\frac{z_i'+qz_j-(q+1)z_i'z_j}{1-z_i'z_j}$	$\frac{q^{-1}z_i'+z_j-(1+q^{-1})z_i'z_j}{1-z_i'z_j}$	$\frac{(1-z_i')(1-qz_j)}{1-z_i'z_j}$	$\frac{(1-q^{-1}z_i')(1-z_j)}{1-z_i'z_j}$

Boltzmann weights for R-matrices

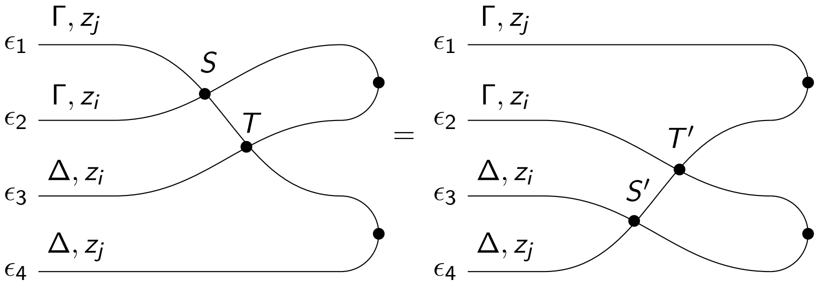
Colored stochastic $\Delta - \Delta$ ice with spectral parameters z_i and z_j ($\alpha < \beta$)



1	1	$\frac{z'_j - z'_i}{q-(q+1)z'_i+z'_i z'_j}$	$\frac{q(z'_j - z'_i)}{q-(q+1)z'_i+z'_i z'_j}$	$\frac{(1-z'_i)(q-z'_j)}{q-(q+1)z'_i+z'_i z'_j}$	$\frac{(1-z'_j)(q-z'_i)}{q-(q+1)z'_i+z'_i z'_j}$

Reflection equation

- As we don't have colored stochastic $\Gamma - \Delta$ ice, we don't have the caduceus relation
- S, T: colored stochastic $\Gamma - \Gamma$, $\Delta - \Gamma$ ice, spectral parameters z_i, z_j
- S', T': colored stochastic $\Delta - \Delta$, $\Delta - \Gamma$ ice, spectral parameters z_j, z_i



Ground state partition function

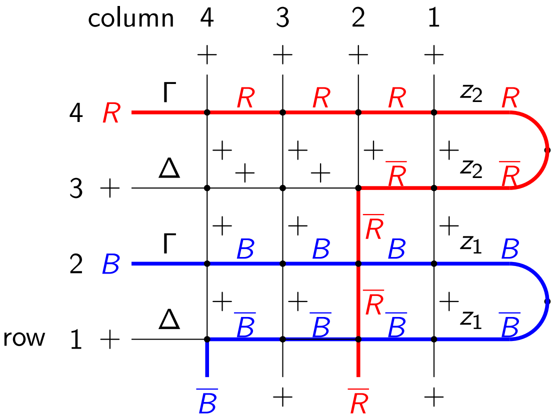
When σ and τ satisfy the condition that $\sigma(i) = \overline{\tau(i)}$ for every $i = 1, \cdots, n$, there is a unique admissible state.

The partition function can be written explicitly.

$$
\begin{aligned}
Z(\mathcal{S}_{n,L,\lambda,\sigma,\tau,z}) &= \prod_{i=1}^{n} z_i^L \prod_{i=1}^{n} (\frac{z_i'}{q})^{\lambda_i+n-i} \prod_{i=1}^{n} (1 - q^{-1_{\sigma(i)<0}} z_i') \\
&\times \quad q^{\sum_{i=1}^{n}(L-n+i+\lambda_i)1_{\sigma(i)>0} + \sum_{1\le i<j\le n}(1_{\overline{\sigma(j)}<\sigma(i)} + 1_{\sigma(j)<\sigma(i)})}.
\end{aligned}
$$

Ground state partition function

$$\overline{R} < \overline{B} < B < R$$



Recursive relation I

Theorem (Z.,2021)

Assume that $1 \leq i \leq n-1$ *and* $\sigma(i+1) > \sigma(i)$, *and let* $s_i = (i, i+1)$. *Then the partition function of the colored model satisfies the following recursive relation:*

$$q^{1_{\sigma(i+1)>0} - 1_{\sigma(i)>0}} Z(\mathcal{S}_{n,L,\lambda,\sigma s_i,\tau,z}) = -A(q,z,i) Z(\mathcal{S}_{n,L,\lambda,\sigma,\tau,z}) + B(q,z,i) Z(\mathcal{S}_{n,L,\lambda,\sigma,\tau,s_i z}),$$

where

$$A(q,z,i) = \frac{(1-z_{i+1})(1-qz_i)}{z_{i+1}-z_i},$$

$$B(q,z,i) = \frac{1-(q+1)z_i + qz_i z_{i+1}}{z_{i+1}-z_i}.$$

Recursive relation II

Theorem (Z.,2021)

Assume that $\sigma(n) > 0$. *Let* s_n *be the element of* B_n *that changes the sign of the element at the nth position, and* $s_n z = (z_1, \cdots, z_{n-1}, \frac{1}{z'_n})$. *Then we have*

$$\begin{aligned}(\frac{q}{z_n})^L Z(\mathcal{S}_{n,L,\lambda,\sigma s_n,\tau,z}) &= C(q,z) z_n^{-L} Z(\mathcal{S}_{n,L,\lambda,\sigma,\tau,z}) \\ &\quad - D(q,z) z_n'^L Z(\mathcal{S}_{n,L,\lambda,\sigma,\tau,s_n z}),\end{aligned}$$

where

$$C(q,z) = \frac{(q - z'_n)(z_n - 1)}{q(1 - z_n z'_n)},$$

$$D(q,z) = \frac{q z_n + z'_n - (q+1) z_n z'_n}{q(1 - z_n z'_n)}.$$

Connection to Demazure-Lusztig operators of type C

- $u = (u_1, \cdots, u_n)$, $f(u)$ any rational function of u
- $s_i u = (u_1, \cdots, u_{i+1}, u_i, \cdots, u_n)$, $1 \le i \le n-1$
- $s_n u = (u_1, \cdots, u_{n-1}, \frac{1}{u_n})$
- $s_i f(u) = f(s_i u)$

Demazure-Lusztig operators of type C:

$$\mathcal{L}_{i,v}(f) = \frac{1-v}{u^{\alpha_i}-1} f + \frac{v u^{\alpha_i}-1}{u^{\alpha_i}-1} s_i(f),$$

where $\alpha_i = \epsilon_i - \epsilon_{i+1}$ for $1 \le i \le n-1$ and $\alpha_n = 2\epsilon_n$. Also let

$$\hat{\mathcal{L}}_{i,v} = \mathcal{L}_{i,v} - v + 1.$$

Connection to Demazure-Lusztig operators of type C

- Let $u_i = \frac{1-qz_i}{1-z_i}$
- Let

$$Z(\tilde{\mathcal{S}}_{n,L,\lambda,\sigma,\tau,u}) = Z(\mathcal{S}_{n,L,\lambda,\sigma,\tau,z}) \prod_{i=1}^{n} z_i^{-L} q^{(n-i)1_{\sigma(i)>0}+(L+1)1_{\sigma(i)<0}}$$

  as a function of u

The recursive relations translate into

- For $1 \leq i \leq n-1$, if $\sigma(i+1) > \sigma(i)$, then

$$Z(\tilde{\mathcal{S}}_{n,L,\lambda,\sigma s_i,\tau,u}) = \hat{\mathcal{L}}_{i,q}(Z(\tilde{\mathcal{S}}_{n,L,\lambda,\sigma,\tau,u})).$$

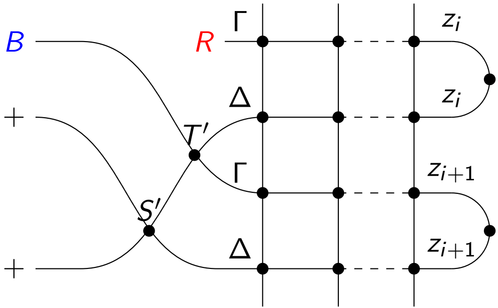
- If $\sigma(n) > 0$, then

$$Z(\tilde{\mathcal{S}}_{n,L,\lambda,\sigma s_n,\tau,u}) = -\mathcal{L}_{n,q}(Z(\tilde{\mathcal{S}}_{n,L,\lambda,\sigma,\tau,u})).$$

Derivation of recursive relation I

Attach two R-vertices to the left.
Assume that $\sigma(i+1) = R$ and $\sigma(i) = B$. Note that $B < R$.

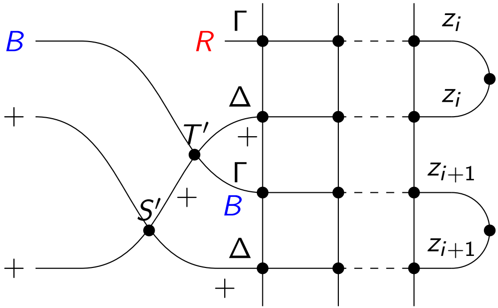
- S': colored stochastic $\Delta - \Delta$ ice, spectral parameters z_i, z_{i+1}
- T': colored stochastic $\Delta - \Gamma$ ice, spectral parameters z_i, z_{i+1}



Derivation of recursive relation I

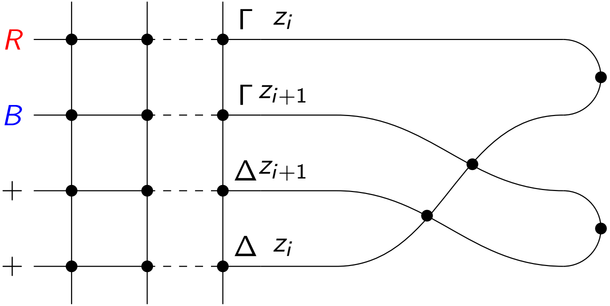
Attach two R-vertices to the left.
Assume that $\sigma(i+1) = R$ and $\sigma(i) = B$. Note that $B < R$.

- S': colored stochastic $\Delta - \Delta$ ice, spectral parameters z_i, z_{i+1}
- T': colored stochastic $\Delta - \Gamma$ ice, spectral parameters z_i, z_{i+1}



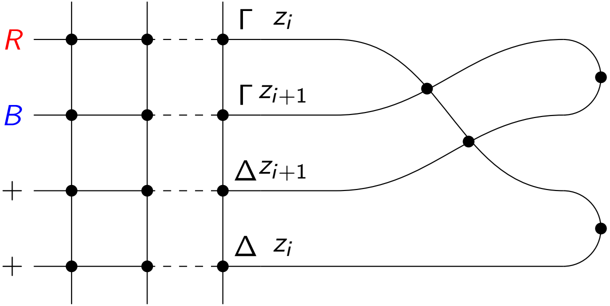
Derivation of recursive relation I

Push the two R-vertices to the right using the Yang-Baxter equations



Derivation of recursive relation I

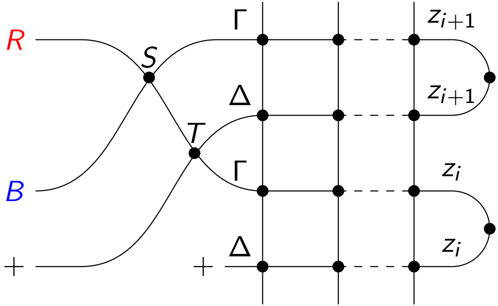
Use the reflection equation to change to the following



Derivation of recursive relation I

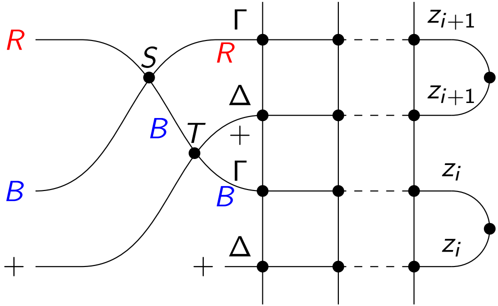
Push the two R-vertices back to the left using the Yang-Baxter equations

- S: colored stochastic $\Gamma - \Gamma$ ice, spectral parameters z_{i+1}, z_i
- T: colored stochastic $\Delta - \Gamma$ ice, spectral parameters z_{i+1}, z_i



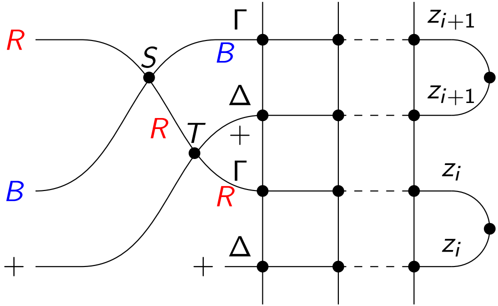
Derivation of recursive relation I

The partition function is the sum of two terms
The first term is



Derivation of recursive relation I

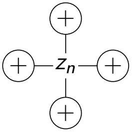
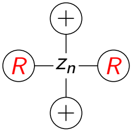
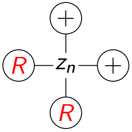
The partition function is the sum of two terms
The second term is



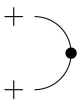
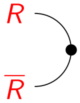
Derivation of recursive relation II

Assume that $\sigma(n) = R$, which is the only possible color in the top row. Top boundary edges only carry the "$+$" spin. Therefore

- Only the following three states may appear in the top row

<table>
<tr><td>
</td><td>
</td><td>
</td></tr>
<tr><td>1</td><td>qz_n</td><td>$1 - qz_n$</td></tr>
</table>

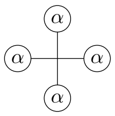
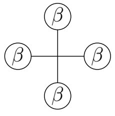
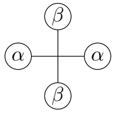
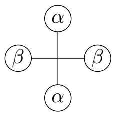
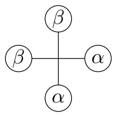
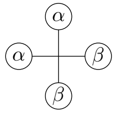
- Only the following two states may appear for the top cap

<table>
<tr><td>Cap</td><td>
</td><td>
</td></tr>
<tr><td>Boltzmann weight</td><td>1</td><td>1</td></tr>
</table>

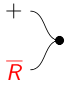
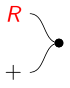
Derivation of recursive relation II

We make the following changes

- Interchange $+$ and R in the top row
- Change the Boltzmann weights of the top row to the following (which is colored stochastic Δ ice with spectral parameter $\frac{1}{z_n'}$), where $\alpha < \beta$:

a_1/a_2	a_1/a_2	b_1	b_2	d_1	d_2
					
1	1	$\frac{1}{z_n}$	$\frac{1}{qz_n}$	$1-\frac{1}{z_n}$	$1-\frac{1}{qz_n}$

- Change the Boltzmann weights of the top cap to the following

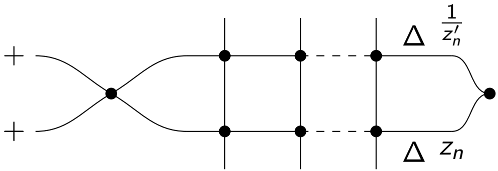
New cap		
Boltzmann weight	1	-1

Derivation of recursive relation II

The partition function of the new system is $(\frac{1}{qz_n})^L$ times the original partition function.

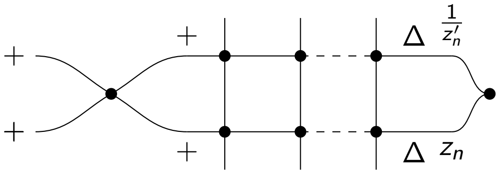
Derivation of recursive relation II

Attach an R-vertex (colored stochastic $\Delta - \Delta$ ice) with spectral parameters $\frac{1}{z_n'}, z_n$ to the left of the top two rows of the new system



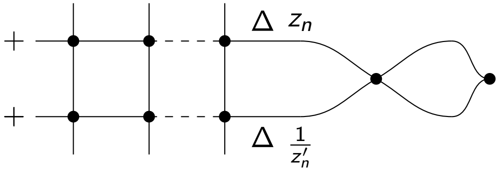
Derivation of recursive relation II

Attach an R-vertex (colored stochastic $\Delta - \Delta$ ice) with spectral parameters $\frac{1}{z_n'}, z_n$ to the left of the top two rows of the new system



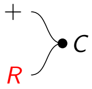
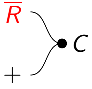
Derivation of recursive relation II

We push the R-vertex to the right using the Yang-Baxter equation

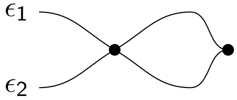


Derivation of recursive relation II

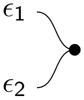
Introduce an auxiliary cap

Auxiliary cap		
Boltzmann weight	-1	1

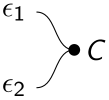
A variant of the fish relation



$$= \frac{(qz_n - 1)(1 - z_n')}{qz_n + z_n' - (q+1)}$$

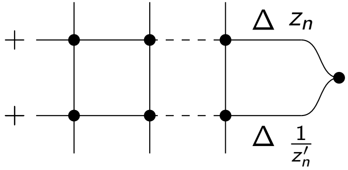


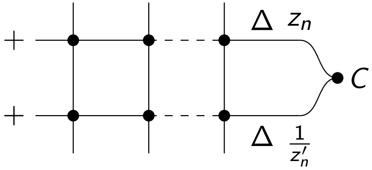
$$+ \frac{q(z_n z_n' - 1)}{qz_n + z_n' - (q+1)}$$



Derivation of recursive relation II

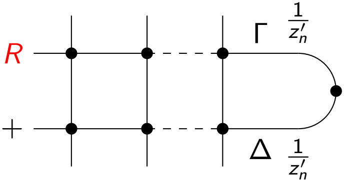
Hence it suffices to compute





Derivation of recursive relation II

For the first one, interchanging R and $+$ in the top row and changing Boltzmann weights accordingly, it can be shown that it is equal to $(\frac{z_n'}{q})^L$ times



Derivation of recursive relation II

For the second one, interchanging $\overline{R}$ and $+$ in the top row and changing Boltzmann weights accordingly, it can be shown that it is equal to $-(z'_n)^L$ times